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# A NOTE ON THE DETERMINANT OF CERTAIN MATRICES

By Laurence R. Walker

In the course of an investigation on electrical circuit theory appearing in this Journal[1] Dr. C. M. Gewertz stated the following theorem as Theorem II, p. 59 of his paper.

"If the determinant of the real parts of the elements of a symmetrical, non-singular square matrix of any order, whose elements are functions of a complex variable, is $\geq 0$, then the determinant of the real parts of the elements of its inverse matrix is also $\geq 0$."

Reference was made to a proof of this theorem due to Dr. P. Franklin which invoked the theory of positive quadratic forms. It is the first purpose of this note to show that a wider theorem dispensing with the symmetry condition may be proved by rather simple considerations.

The theorem to be proved here reads as follows:

"The determinant of the real parts of the elements of a square, non-singular matrix with complex elements has the same sign as the determinant of the real parts of the elements of its inverse matrix, while the determinant of the complex parts of the matrix has the same or the opposite sign to the determinant of the complex parts of the inverse matrix according as the degree of the matrix is even or odd; if either of the determinants of the matrix be zero, so is the corresponding determinant of the inverse."

Let

$$C + iD = (A + iB)^{-1}$$

Then

$$(C + iD)(A + iB) = E \tag{1}$$

where $E$ will be used for the unit matrix.

This entails

$$CA - DB = E \tag{2}$$

$$CB + DA = 0 \tag{3}$$

[1] C. M. Gewertz, Synthesis of a finite, four-terminal network, Jour. Math. and Phys., V. 12, p. 1.



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From (3), provided that $\det A \neq 0$, we have

$$D = -CBA^{-1} \tag{4}$$

Substituting in (2), one finds

$$CA + CBA^{-1}B = E \tag{5}$$

$$CA(E + A^{-1}B \cdot A^{-1}B) = E \tag{6}$$

Thus,

$$\det C \cdot \det A \cdot \det (E + A^{-1}B \cdot A^{-1}B) = 1 \tag{7}$$

Now,

$$\det (E + A^{-1}B \cdot A^{-1}B) = \det (E + iA^{-1}B) \cdot \det (E - iA^{-1}B) \tag{8}$$

and since the elements of $A^{-1}B$ are real the two determinants on the right hand side must have conjugate complex elements and thus conjugate complex values. Thus,

$$\det (E + A^{-1}B \cdot A^{-1}B) > 0 \tag{9}$$

and

$$\det C . \det A > 0 \tag{10}$$

Similarly, if $\det B \neq 0$

$$C = -DAB^{-1} \tag{11}$$

$$-DAB^{-1}A - DB = E \tag{12}$$

$$-DB(B^{-1}A \cdot B^{-1}A + E) = E \tag{13}$$

$$\det D \cdot \det B \cdot \det (E + B^{-1}A \cdot B^{-1}A) = \det (-E) = (-1)^n \tag{14}$$

Then, again, since $\det (E + B^{-1}A \cdot B^{-1}A) > 0$, $\det B . \det D$ has the sign of $(-1)^n$, where $n$ is the order of the matrix. The singular cases, $\det A = 0$, $\det B = 0$, can only lead to $\det C = 0$, $\det D = 0$, respectively, since the theorem as proved so far shows conversely that if $\det C \neq 0$ or $\det D \neq 0$ then $\det A \neq 0$, $\det B \neq 0$, respectively.

It may be noted that if $\det (B - \lambda_r A) = 0$, or, equally, if the $\lambda_r$ are the characteristic roots of $BA^{-1}$ or $A^{-1}B$, when $\det A \neq 0$, we have

$$\det (E + A^{-1}B \cdot A^{-1}B) = \prod_{r=1}^{n} (1 + \lambda_r^2) \tag{15}$$

and, evidently,

$$\det C \cdot \det A = \prod_{r=1}^{n} (1 + \lambda_r^2)^{-1} \tag{16}$$

$$\det D \cdot \det B = (-1)^n \prod_{r=1}^{n} \left( \frac{\lambda_r^2}{1 + \lambda_r^2} \right) \tag{17}$$

In this proof, the fact that $\det (E + X^2) > 0$, where the elements of $X$ are real, has been used. Dr. Franklin has pointed out that this raises the question whether any similar theorem holds for the determinants of matrices of the form $\sum_i X_i^2$, where the elements of all $X_i$ are real. It may, in fact, be shown that if the $X_i$ be a set of $m$ square matrices of order $n$, each of which commutes with a matrix $X_0$, the elements of all the matrices being real, then $\sum_{i=0}^{m} X_i^2$ has a positive determinant.

We have, by our hypothesis,

$$X_0 X_i = X_i X_0 \tag{18}$$

Now, we know that there will exist $n$ one-rowed matrices $S_a$ (some of which may be linearly dependent), which we may call characteristic vectors of $X_0$ such that

$$X_0 S_a = a S_a \tag{19}$$

where $a$ is a scalar.

Thus,

$$\begin{aligned} X_i X_0 S_a &= a X_i S_a \\ &= X_0 X_i S_a \end{aligned} \tag{20}$$

from which it follows that $X_i S_a$ is a linear combination of the $S_a$ corresponding to the characteristic root $a$. We may now choose linear combinations, $S_{a1}$, of the $S_a$ which will also be characteristic vectors of $X_1$. We may continue to find linear combinations, $S_{a2}$, of these which will satisfy $X_2 S_{a2} = a_2 S_{a2}$. Proceeding in this way through the $X_i$, we finally reach a set of $n$ vectors $T_s = R_s + iI_s$, where $R_s$ and $I_s$ are real, which are simultaneously characteristic vectors of $X_0$ and all the $X_i$, corresponding to the characteristic roots $p_{0s} + iq_{0s}$ and $p_{is} + iq_{is}$, respectively.

Thus, we have

$$X_i(R_s + iI_s) = (p_{is} + iq_{is})(R_s + iI_s) \qquad i = 0, 1, 2, \cdots, m \tag{21}$$

This implies, since (21) is equivalent to a set of linear equations and the elements of the $X$ are real,

$$X_i(R_s - iI_s) = (p_{is} - iq_{is})(R_s - iI_s) \qquad i = 0, 1, 2, \cdots, m \tag{22}$$

If $q_{is} = 0$, for any one of the $i$, we have

$$X_i(R_s + iI_s) = p_{is}(R_s + iI_s) \tag{23}$$

and we may choose $I_s = 0$. This plainly gives

$$q_{js} = 0 \qquad j \neq i \tag{24}$$

Thus, if $F = \sum_{i=0}^{m} X_i^2$, we have

$$FT_s = \sum_{i=0}^{m} X_i^2 T_s = T_s \sum_{i=0}^{m} (p_{is} + iq_{is})^2 \tag{25}$$

and the characteristic values of $F$ will be $\sum_{i=0}^{m} (p_{is} + iq_{is})^2$. From (21), (22), (24) we see that either all the terms in one of these sums are real, making the sum positive or the sums may be grouped in pairs, the corresponding terms of the two sums being conjugate complexes. Thus the roots of $F$ are either positive or occur in conjugate complex pairs. Finally, since the determinant of $F$ is the product of its characteristic roots it must be always positive.

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# ON THE STEADY TWO-DIMENSIONAL MOTION OF A VISCOUS LIQUID PAST A FIXED CIRCULAR CYLINDER

By B. A. Griffith

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1. **Introduction.** The determination of the steady flow of a viscous liquid past a fixed solid is a problem which has attracted considerable attention in hydrodynamics. The difficulties encountered in attempting to solve such a problem are well-known and are due to the fact that the equations of motion are non-linear partial differential equations. Thus it is not surprising that all the known solutions have been obtained by methods of approximation which appear to be justified provided the Reynolds number is sufficiently small. In the present paper a new method of approximation is proposed in which the methods of Stokes[1] and Oseen[2] are combined. The method may be applied to any case for which the Oseen solution is known but the analytical details depend on the form of the solid. Consequently, to avoid confusion, we shall confine our attention to a special case of the above problem, viz.: the two-dimensional flow past a fixed, right-circular cylinder.

[1] Sir H. Lamb, Hydrodynamics, (Cambridge, 1932), §338.
[2] Sir H. Lamb, op. cit., §342

Referred to fixed rectangular axes $Oxy$ the exact equations for the steady two-dimensional flow of a viscous liquid under no body forces are

$$\text{(1.1)} \qquad \begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} + \nu\Delta u, \\ u\dfrac{\partial v}{\partial x} + v\dfrac{\partial v}{\partial y} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial y} + \nu\Delta v, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases}$$

where $(u, v)$ are the velocity components and $p$ is the pressure at a point $(x, y)$ in the plane of the motion. The constants $\rho$ and $\nu$ appearing in these equations, denote respectively the density and the kinematic coefficient of viscosity of the liquid; the operator $\Delta$ is the usual Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2$.

It is our purpose to find a steady motion (that is, to obtain expressions for $u$, $v$ and $p$ as continuous functions of position) which approximates to a solution of the equations (1.1) and which satisfies the boundary conditions required for flow past a fixed, right-circular cylinder of radius $a$, whose axis is at right angles to the plane of the motion. Of course it is necessary to assume that the cylinder is infinite in length in order that the motion may be two-dimensional. In order to simplify the boundary conditions we shall suppose the liquid to be infinite in extent and shall take special axes $Oxy$ such that $O$ lies in the axis of the cylinder and $Ox$ is parallel to the constant velocity of the stream at infinity. We may then write the boundary conditions of our problem in the following form:

$$\text{(1.2)} \qquad \begin{cases} u = v = 0 \quad \text{at} \quad r \equiv (x^2 + y^2)^{1/2} = a; \\ \lim\limits_{r\to\infty} u = U, \qquad \lim\limits_{r\to\infty} v = 0, \end{cases}$$

where $U$ is the constant magnitude of the velocity at infinity.

It has been stated that the method of approximation proposed in this paper is based on a combination of the methods of Stokes and Oseen. Now the method of Stokes consists in ignoring the non-linear terms in the exact equations of motion. Hence the Stokes equations for the motion under consideration are

$$(1.3)\qquad \begin{cases} \dfrac{\partial p}{\partial x} = \mu \Delta u, \\ \dfrac{\partial p}{\partial y} = \mu \Delta v, \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0, \end{cases}$$

where $\mu (= \rho\nu)$ is the ordinary coefficient of viscosity of the liquid.

A solution of these equations for $u$, $v$ and $p$ may be said to approximate to a solution of the exact equations (1.1) at all points in the liquid where the terms in (1.1), which are neglected in arriving at the equations (1.3), are small in comparison with the terms retained. This would certainly seem to be true at points near the surface of the cylinder, on account of the assumed continuity of velocity and the fact that $u$ and $v$ must vanish at $r = a$. The approximation is evidently less satisfactory at points remote from the cylinder, but even for such points it might be hoped that the approximation could be justified provided the Reynolds number $R (= 2Ua/\nu)$ is sufficiently small. Unfortunately this does not prove to be the case and it is found impossible to solve the equations (1.3) subject to the boundary conditions (1.2) save in the trivial case where $U$, and consequently $R$, vanishes. This well-known result was first obtained by Stokes and is usually referred to as Stokes' paradox.[3]

In 1910 Oseen[4] proposed an alternative method of approximating to the exact equations of motion for a viscous liquid. For the problem under consideration this method is equivalent to the following. Writing

$$(1.4)\qquad u = U + u', \qquad v = v', \qquad p = p',$$

we assume that $u'$, $v'$ are everywhere small if $R$ is small. Squares and products of $u'$, $v'$ and their derivatives are then neglected when substituting for $u$, $v$ from (1.4) in the equations (1.1). In this way we obtain the following approximate equations of Oseen:

$$(1.5)\qquad \begin{cases} U \dfrac{\partial u'}{\partial x} = -\dfrac{1}{\rho} \dfrac{\partial p'}{\partial x} + \nu \Delta u', \\ U \dfrac{\partial v'}{\partial x} = -\dfrac{1}{\rho} \dfrac{\partial p'}{\partial y} + \nu \Delta v', \\ \dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} = 0. \end{cases}$$

[3] Sir H. Lamb, op. cit., p. 614.

[4] C. W. Oseen, *Ueber die Stokes'che Formel, und über eine verwandte Aufgabe in der Hydrodynamik*, Arkiv för matematik och fysik, 6 (1910), no. 29.

These equations are to be solved for $u'$, $v'$, and $p'$ as functions of position subject to the following boundary conditions on $u'$ and $v'$:

$$\begin{cases} u' = -U, \qquad v' = 0 \quad \text{at} \quad r = a; \\ \lim_{r\to\infty} u' = \lim_{r\to\infty} v' = 0. \end{cases} \tag{1.6}$$

It is clear that for points remote from the cylinder the terms in (1.1), which are neglected in arriving at the equations (1.5), are smaller than the terms retained. However, for points in the vicinity of $r = a$, $u'$ approximates to the value $-U$ and consequently, for such points, the terms $u'\partial u'/\partial x$ and $u'\partial v'/\partial x$, which are neglected, appear to be of the same order as the terms retained in (1.5). Thus the approximation appears to be satisfactory for points remote from the cylinder but the same cannot be said for points in the immediate neighbourhood of $r = a$. However, in spite of this defect, Oseen's method of approximation seems to be valid provided $R$ is sufficiently small.

In 1911 Lamb[5] succeeded in finding a solution of the equations (1.5) which satisfies the second set of conditions in (1.6) exactly, but which can only be made to satisfy the first set if terms of the order $R^2$ are neglected. A more general solution of the equations (1.5) was obtained by Bairstow[6] in 1923. The expressions for $u'$, $v'$ and $p'$ corresponding to this solution are infinite series and if we retain a sufficient number of terms in these series it is possible to satisfy, to any desired degree of approximation, the boundary conditions required for flow past a cylinder of any cross-section. Bairstow applied this solution to the case of a circular cylinder and obtained numerical values for the drag coefficient corresponding to several values of $R$. These results were compared with the experimental results found by Relf[7] and the comparison shows that there is fairly good agreement between the calculated and observed values of the drag for values of $R$ less than 5. As $R$ increases beyond 5 the calculated results deviate from the observed results and at $R = 25$ Bairstow finds a value for the drag which is 70 per cent higher than the value observed by Relf.

[5] Sir H. Lamb, *On the Uniform Motion of a Sphere through a Viscous Fluid.* Phil. Mag. (6) 21 (1911), 112–121.

[6] L. Bairstow, Miss B. M. Cave, and Miss E. D. Lang, *The Resistance of a Cylinder Moving in a Viscous Fluid.* Phil. Trans. Roy. Soc. A, 223 (1923), 383–432.

[7] E. F. Relf, *Discussion of the Measurements on the Resistance of Wires with some Additional Tests on the Resistance of Wires of Small Diameter.* Adv. Comm. Aeronautics, Rept. and Mem. 102 (1914).

A solution of the equations (1.5) which may be made to satisfy the boundary conditions (1.6) exactly has been given by Filon[8] and also by Faxén.[9] In this solution the expressions for $u'$, $v'$ and $p'$ are infinite series whose convergence has been investigated by Faxén in the paper referred to. Although these writers do not apply this solution to obtain numerical results for the drag coefficient the results of such a calculation are substantially in agreement with the results obtained by Bairstow.

In view of the discrepancy between the experimental values for the drag and those predicted by Oseen's method when $R$ exceeds 5 it appears necessary to seek some means of improving the approximation. In 1928 Filon[10] sought to extend the Oseen method but was led to the conclusion that the Oseen solution is not a sufficiently good approximation to the solution of the exact equations (1.1) to serve as a basis for a method of successive approximations. Again, in 1933 Southwell and Squire[11] proposed an alternative method in which the convection of vorticity is ascribed to an irrotational motion rather than to the undisturbed stream as in Oseen's method. Their results for the drag, while in better agreement with experiment than those predicted by Oseen's method are still much too high. In addition to these analytical methods there is the numerical method used by Thom,[12] which, although it leads to excellent results, is necessarily a method which must be repeated for each different value of the Reynolds number.

The method introduced in this paper is based on the following hypothesis: for points near the cylinder the equations (1.3) are a better approximation to the equations (1.1) than are the equations (1.5), and the reverse is the case for points remote from the cylinder. Accordingly we seek: (1) a solution of the equations (1.3), valid within a region bounded by the cylinders $r = a$ and $r = X$, where $X > a$, which solution satisfies the first set of boundary conditions in (1.2); (2) a solution of the

[8] L. N. G. Filon, *The Force on a Cylinder in a Stream of Viscous Fluid.* Proc. Roy. Soc. A, 113 (1926), 7-27.

[9] H. Faxén, *Exakte Losung der Oseenschen Differentialgleichungen einer zähen Flüssigkeit für den Fall der Translationsbewegung eines Zylinders.* Nova Acta, Reg. Soc. Sci. Upsaliensis, Volumen extra ordinem editum (1927).

[10] L. N. G. Filon, *On the Second Approximation to the "Oseen" Solution for the motion of a Viscous Fluid.* Phil. Trans. Roy. Soc., A, 227 (1928), 93-135.

[11] R. V. Southwell and H. B. Squire, *A Modification of Oseen's Approximate Equations for the Motion in Two Dimensions of a Viscous Incompressible Fluid.* Phil. Trans. Roy. Soc. A, 232 (1933), 27-64.

[12] A. Thom, *Flow Past Circular Cylinders at Low Speeds.* Proc. Roy. Soc. A, 141 (1933), 651-669.

equations (1.5), valid for all points outside the cylinder $r = X$ and satisfying the second set of boundary conditions in (1.6). When these solutions have been found we may fit them together over the surface $r = X$ by insisting that the components of the velocity and stress vectors be continuous.

The method is applied to find values for the drag coefficient corresponding to more than 40 pairs of values for $R$ and the ratio $\xi(= X/a)$, and the results are compared with those predicted by Oseen's method and also with the experimental values observed by Wieselsberger[13] and others. Since the resulting values for the drag depend on the artificial parameter $\xi$ in addition to the natural parameter $R$, it may be argued that the method would naturally be expected to give agreement with experiment provided a proper choice of $\xi$ is made corresponding to each value of $R$. The truth of this argument must be admitted. However the method leads to results for the drag which agree with those predicted by Oseen's method when $\xi = 1$. As $\xi$ is increased beyond unity the values of the drag coefficient decrease and ultimately lie below the experimental values. Consequently it appears that we should attach to the results obtained by our method for a fixed value of $\xi$ just as much importance as we do to the results obtained by Oseen's method with the advantage that when $\xi$ is not too large the results agree more closely with experiment.

**2. The solution of Stokes' equations.** In order to determine the motion within the region $a \leq r \leq X$ we must first solve the equations (1.3) subject to the first set of conditions in (1.2). To do this we proceed in the following manner. Multiplying the second equation in (1.3) by $i = \sqrt{-1}$ and adding the result to the first equation we have the equation

$$Dp = \mu DD^*w, \tag{2.1}$$

where

$$D = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y},$$

$$D^* = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y},$$

$$w = u + iv.$$

[13] C. Wieselsberger, Physik. Z., 22 (1921), 321–328.

Again, by making use of the third equation in (1.3), we have

$$D^*w = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$
$$= 2i\omega,$$

where $\omega$ is the vorticity. Thus the equations

$$D(p - 2\mu i\omega) = 0, \qquad D^*w = 2i\omega, \tag{2.2}$$

are equivalent to the equations (1.3).

The first equation in (2.2) is at once recognized as the necessary and sufficient condition that

$$p - 2\mu i\omega = F(z),$$

where $F(z)$ is an analytic function of $z = x + iy$. Now, since $p$ and $\omega$ are single-valued functions of position, we may write

$$p - 2\mu i\omega = \sum_{n=-\infty}^{\infty} a_n \frac{r^n}{X^{n+1}} e^{in\theta}, \tag{2.3}$$

where $(r, \theta)$ are polar coordinates in the plane of motion defined by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The constants $a_n$ which appear in the expansion on the right hand side of the equation (2.3) are necessarily real on account of the symmetry of the problem with respect to the line $\theta = 0$.

To complete the solution of Stokes' equations we must determine $w$ to satisfy the second equation in (2.2). In order to do this, we assume the following expansion for $w$:

$$w = \sum_{n=-\infty}^{\infty} f_n(r) e^{in\theta}, \tag{2.4}$$

where $f_n(r)$ must be real functions on account of symmetry. It is assumed that this expansion for $w$ is convergent and admits term by term differentiation. Then

$$D^*w = \left[\left(\cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}\right) - i\left(\sin\theta\,\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\,\frac{\partial}{\partial\theta}\right)\right]w$$

$$= (\cos\theta - i\sin\theta)\left(\frac{\partial}{\partial r} - \frac{i}{r}\,\frac{\partial}{\partial\theta}\right)w$$

$$= e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\,\frac{\partial}{\partial\theta}\right)\sum_{n=-\infty}^{\infty} f_n(r)\,e^{in\theta}$$

$$= \sum_{n=-\infty}^{\infty}\left(\frac{df_n}{dr} + \frac{n}{r}\,f_n\right)e^{i(n-1)\theta}.$$

But from (2.3) we have

$$2i\omega = -\frac{1}{\mu}\sum_{n=-\infty}^{\infty} ia_n\,\frac{r^n}{X^{n+1}}\sin n\theta$$

$$= -\frac{1}{2\mu}\sum_{n=-\infty}^{\infty} a_n\,\frac{r^n}{X^{n+1}}\,(e^{in\theta} - e^{-in\theta}).$$

Thus the value of $w$ given by (2.4) satisfies the second equation in (2.2) provided

$$(2.5)\quad r\,\frac{df_n}{dr} + nf_n = -\frac{1}{2\mu}\left[a_{n-1}\,\frac{r^n}{X^n} - a_{1-n}\,\frac{r^{2-n}}{X^{2-n}}\right],\quad (n = 0, \pm1, \pm2, \cdots).$$

By writing $\log_e r = t$, we see that we may write the equations (2.5) in the following form:

$$\frac{df_n}{dt} + nf_n = -\frac{e^{-nt}}{2\mu}\left[a_{n-1}\,\frac{e^{2nt}}{X^n} - a_{1-n}\,\frac{e^{2t}}{X^{2-n}}\right],\quad (n = 0, \pm1, \pm2, \cdots).$$

The general solution of these equations is easily seen to be

$$f_n = e^{-nt}\left[c_n - \frac{1}{2\mu}\int\left(a_{n-1}\,\frac{e^{2nt}}{X^n} - a_{1-n}\,\frac{e^{2t}}{X^{2-n}}\right)dt\right],\quad (n = 0, \pm1, \pm2, \cdots),$$

where $c_n$ are constants. Thus, we have

$$(2.6)\qquad \begin{cases} f_0 = c_0 - \dfrac{1}{2\mu}\left(a_{-1}\log_e r - \dfrac{a_1}{2}\,\dfrac{r^2}{X^2}\right), \\ r^n f_n = c_n - \dfrac{1}{2\mu}\left(\dfrac{a_{n-1}}{2n}\,\dfrac{r^{2n}}{X^n} - \dfrac{a_{1-n}}{2}\,\dfrac{r^2}{X^{2-n}}\right),\quad (n = \pm1, \pm2, \cdots). \end{cases}$$

The solution of the equations (2.2), given by (2.3) and (2.4), will clearly satisfy the first set of conditions in (1.2) provided

$$f_n(a) = 0,\qquad (n = 0, \pm1, \pm2, \cdots).$$

Then from (2.6) we have

$$c_0 = \frac{1}{2\mu}\left(a_{-1}\log_e a - \frac{a_1}{2}\frac{a^2}{X^2}\right),$$

$$c_n = \frac{1}{2\mu}\left(\frac{a_{n-1}}{2n}\frac{a^{2n}}{X^n} - \frac{a_{1-n}}{2}\frac{a^2}{X^{2-n}}\right), \quad (n = \pm 1, \pm 2, \cdots).$$

The required solution of the equations (1.3) subject to the first set of conditions in (1.2) is thus given by

$$(2.7)\qquad \begin{cases} w = u + iv = \sum_{n=-\infty}^{\infty} f_n(r)e^{in\theta}, \\ p - 2\mu i\omega = \sum_{n=-\infty}^{\infty} a_n \frac{r^n}{X^{n+1}} e^{in\theta}, \end{cases}$$

where

$$f_0 = -\frac{1}{2\mu}\left(a_{-1}\log_e \frac{r}{a} - \frac{a_1}{2}\frac{r^2 - a^2}{X^2}\right),$$

$$(2.8)\qquad r^n f_n = -\frac{1}{2\mu}\left(\frac{a_{n-1}}{2n}\frac{r^{2n} - a^{2n}}{X^n} - \frac{a_{1-n}}{2}\frac{r^2 - a^2}{X^{2-n}}\right),$$

$$(n = \pm 1, \pm 2, \cdots).$$

3. **The solution of Oseen's equations.** The solution given below for Oseen's equations (1.5) agrees formally with that given by Filon[14] and subsequently by Faxén[15] although the method by which it is obtained is somewhat different. The method used here is, in fact, a close parallel to the method used in (2) to solve Stokes' equations. Thus we multiply the second equation in (1.5) by $i$ and add the result to the first equation, obtaining the equation

$$(3.1)\qquad \rho U\frac{\partial w'}{\partial x} = -Dp' + \mu\Delta w',$$

where $w' = u' + iv'$, and $D$ is the operator $\partial/\partial x + i\partial/\partial y$ as in (2). Now from (1.5) it follows that $\Delta p' = 0$ and hence we may write

$$(3.2)\qquad p' = C_0 + \rho U\phi,$$

[14] Loc. cit.
[15] Loc. cit.

where $C_0$ is a constant and $\phi$ is a solution of $\Delta\phi = 0$. Introducing the function $\psi$, the harmonic conjugate to $\phi$, so that

$$\Omega = \phi + i\psi \tag{3.3}$$

is an analytic function of $z = x + iy$, we may also write

$$w' = -\Omega^* + w'', \tag{3.4}$$

where $\Omega^*$ denotes the conjugate of $\Omega$. Making the substitution from (3.2) and (3.4) in (3.1) we find the following equation for $w''$:

$$\rho U \frac{\partial w''}{\partial x} = \mu \Delta w''. \tag{3.5}$$

Writing

$$w'' = e^{\sigma z}\varpi, \tag{3.6}$$

where $\sigma = \rho U/2\mu$, we see from (3.5) that $\varpi$ satisfies the equation

$$(\Delta - \sigma^2)\varpi = 0. \tag{3.7}$$

Since the equation (3.1) replaces the first two equations of (1.5) it follows that $p'$ and $w' = u' + iv'$, as given by (3.2) and (3.4), constitute a solution of Oseen's equations provided $w''$ has the value given by (3.6), where $\varpi$ is a solution of (3.7), and if in addition

$$\begin{aligned} D^*w' = \frac{\partial w'}{\partial x} - i\frac{\partial w'}{\partial y} &= \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} + i\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial x}\right) \\ &= 2i\omega', \end{aligned} \tag{3.8}$$

where $\omega'$ is the vorticity. But

$$D_i^*w' = D^*w'' = e^{\sigma z}(\sigma\varpi + D^*\varpi), \tag{3.9}$$

since $D^*\Omega^* = 0$. Thus, in order to complete the solution of the equations (1.5), we must choose $\Omega$ properly and determine $\varpi$ as a solution of (3.7) such that

$$R\{\sigma\varpi + D^*\varpi\} = 0, \tag{3.10}$$

where $R\{\ \}$ denotes the real part of the quantity within the brackets.

On account of the boundary conditions at infinity it may be seen that

$\Omega$ must approach zero as $r$ tends to infinity and consequently we may assume for $\Omega$ the expansion

$$\Omega = \sum_{n=1}^{\infty} C_n X^n z^{-n} = \sum_{n=1}^{\infty} C_n X^n r^{-n} e^{-in\theta}, \tag{3.11}$$

where the constants $C_n$ are real. Using this expansion for $\Omega$ we see that $p'$ and $\Omega^*$ have the expansions

$$\begin{cases} p' = C_0 + \rho U \sum_{n=1}^{\infty} C_n X^n r^{-n} \cos n\theta, \\ \Omega^* = \sum_{n=1}^{\infty} C_n X^n r^{-n} e^{in\theta}. \end{cases} \tag{3.12}$$

In order to find a solution for $\varpi$ we assume

$$\varpi = U \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta}, \tag{3.13}$$

where, on account of symmetry, the functions $g_n(r)$ are real. It will be further assumed that this expansion for $\varpi$ admits term by term differentiation as often as required for substitution in the equation (3.7). Carrying out this substitution we find the following equations for the functions $g_n(r)$:

$$\frac{d^2 g_n}{dr^2} + \frac{1}{r}\frac{dg_n}{dr} - \left(\frac{n^2}{r^2} + \sigma^2\right) g_n = 0, \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{3.14}$$

If we introduce the independent variable $\sigma r$ in place of $r$ we find, for a fixed value of $n$, that $g_n$ satisfies the modified Bessel equation of order $n$. Seeing that we require $\varpi = 0$ at infinity, the general solution of (3.14), suitable for our purpose, is

$$g_n = B_n K_n(\sigma r), \qquad (n = 0, \pm 1, \pm 2, \cdots), \tag{3.15}$$

where the constants $B_n$ are real and $K_n(\sigma r)$ denotes the modified Bessel function of the second kind[16] and order $n$.

It remains to show that this solution for $\varpi$ may be made to satisfy the equation (3.10). Using the value of $\varpi$ given by (3.13), where $g_n(r)$ have the values given by (3.15), we find

[16] G. N. Watson, *Theory of Bessel Functions*, (Cambridge, 1922), p. 78.

$$\sigma\bar{w} + D^*\bar{w} = \left[\sigma + e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right)\right] U \sum_{n=-\infty}^{\infty} B_n K_n(\sigma r) e^{in\theta}$$

$$= U \sum_{n=-\infty}^{\infty} B_n \left[\sigma K_n(\sigma r) + e^{-i\theta}\left(\sigma \frac{dK_n(\sigma r)}{d(\sigma r)} + \frac{n}{r} K_n(\sigma r)\right]\right] e^{in\theta}$$

$$= \sigma U \sum_{n=-\infty}^{\infty} \left[B_n K_n(\sigma r) + B_{n+1}\left(\frac{dK_{n+1}(\sigma r)}{d(\sigma r)} + \frac{n+1}{\sigma r} K_{n+1}(\sigma r)\right)\right] e^{in\theta} \tag{3.16}$$

$$= \sigma U \sum_{n=-\infty}^{\infty} (B_n - B_{n+1}) K_n(\sigma r) e^{in\theta},$$

since[17]

$$z \frac{dK_n(z)}{dz} + nK_n(z) = -zK_{n-1}(z).$$

Then equation (3.10) is satisfied by our solution for $\bar{w}$ provided

$$(B_n - B_{n+1})K_n(\sigma r) + (B_{-n} - B_{1-n})K_{-n}(\sigma r) = 0, \qquad (n = 0, 1, 2, \cdots). \tag{3.17}$$

If we make use of the fact that

$$K_n(\sigma r) = K_{-n}(\sigma r)$$

it follows easily from the equations (3.17) that

$$B_{-n} = B_{n+1}, \qquad (n = 0, 1, 2, \cdots). \tag{3.18}$$

We may now write our solution of the equations (1.5) in the following final form:

$$\begin{cases} w' = u' + iv' = -\sum_{n=1}^{\infty} C_n X^n r^{-n} e^{in\theta} + Ue^{\sigma z} \sum_{n=-\infty}^{\infty} B_n K_n(\sigma r) e^{in\theta}, \\ p' = C_0 + \rho U \sum_{n=1}^{\infty} C_n X^n r^{-n} \cos n\theta, \end{cases} \tag{3.19}$$

where $C_n$, $B_n$ are real constants satisfying the relations (3.18). It should be noted here that these expressions only constitute a solution of the equations (1.5) provided the series involved converge and admit the requisite term by term differentiations. Obviously this will not be

[17] G. N. Watson, op. cit., p. 79.

true for all values of the constants $C_n$, $B_n$. In the next section a number of conditions are given which, together with (3.18), suffice for the determination of $C_n$, $B_n$, and also the constants $a_n$ appearing in the solution of Stokes' equations. If we find that these conditions give values for $a_n$, $C_n$, and $B_n$ such that the series in (2.7) and (3.19) converge properly, then, and then only, can we say that our solution is satisfactory. We note that if the series in (3.19) converge properly the solution of (1.5) corresponds to a motion, symmetrical about the line $\theta = 0$, and satisfying the second set of conditions in (1.6) as required.

**4. On the method of fitting together the solutions of Stokes' and Oseen's equations.** It has been stated that the solutions of Stokes' and Oseen's equations are to be fitted together over the surface $r = X$ by insisting that the velocity and stress vectors be continuous. Now by the term *stress vector* we understand the stress across a surface in the liquid at a point of this surface (in this case the surface $r = X$) and we must not confuse this quantity with the *stress* in the liquid which is a tensor whose components are defined at a point and do not depend on any direction at that point. The usual way of stating the continuity conditions at a surface in a viscous liquid is to insist that velocity and stress be continuous but it is evident that all the physical conditions of the problem demand is continuity of the velocity and stress vectors. The fact that these two statements of the continuity conditions at a surface in a viscous liquid are equivalent may be obvious but the author has not seen this result explicitly stated in the literature. At any rate this result is contained in the following theorem which may be of some interest.

THEOREM. *For the motion of an incompressible viscous fluid the continuity, across a surface S in the fluid, of the quantities in any one of the following groups*: (*a*) *the components of the velocity and stress vectors*; (*b*) *velocity, pressure, and vorticity*; (*c*) *velocity, pressure, and the first partial derivatives of velocity*; (*d*) *velocity and stress*; *implies the continuity of the quantities in the remaining groups.*

In the following proof of the above theorem we deal with a liquid moving in three dimensions. The corresponding theorem for two-dimensional motion follows as a particular case.

The continuity of the quantities in (a), (b), and (d) follows immediately from the continuity of the quantities in (c). Furthermore, the continuity of the quantities in (a) follows at once from the continuity of those in (d). Hence the theorem will be proved if we can show that

the continuity of the quantities in (a) or (b) implies the continuity of those in (c).

In order to see that the continuity of the quantities in (c) follows from the continuity of those in (a) we note that from the continuity of the latter, and the equation of continuity for a liquid, we have the continuity of seven independent quantities viz. the three components of velocity, the three components of the stress vector, and the sum of three first partial derivatives of velocity. In addition, since the three components of velocity are continuous across $S$, it follows that a partial derivative of any one of these components corresponding to a direction lying in the tangent plane to $S$, at the point in question, is continuous across $S$. This gives us six additional independent continuity conditions which, with the seven mentioned above, are sufficient to insure the continuity across $S$ of the thirteen quantities in (c).

Similarly, from the continuity of the quantities in (b) and the equation of continuity we have the continuity across $S$ of eight independent quantities. The continuity of these quantities together with that of the six derivatives of velocity corresponding to two directions at right angles in the tangent plane to $S$, at the point in question, give altogether thirteen independent continuity conditions which suffice to establish the continuity of the quantities in (c). The theorem is now completely established.

From the above theorem it follows that the continuity conditions at $r = X$ are satisfied if we make velocity, pressure, and vorticity continuous across $r = X$. Thus, if we use the subscript '$o$', to denote the value of the corresponding quantity evaluated at a point on $r = X$, it follows that we may write the conditions at $r = X$ in the form

$$(4.1) \qquad \begin{cases} w_0 = u_0 + iv_0 = w_0' + U = U + u_0' + iv_0', \\ p_0 - 2\mu i\omega_0 = p_0' - 2\mu i\omega_0'. \end{cases}$$

It will subsequently be found that these conditions together with (3.18) suffice for the determination of the constants $a_n$, $B_n$, and $C_n$ appearing in our solutions of Stokes' and Oseen's equations.

5. **Deduction of the fundamental equations.** In order to obtain the explicit equations for the constants $a_n$, $B_n$, and $C_n$ appearing in the solutions (2.7) and (3.19), we must substitute in (4.1) for $w_0$, $w_0'$, $p_0$, $p_0'$, $\omega_0$ and $\omega_0'$ the values found from these solutions. Before doing this it is necessary to find from (3.19) series expressions for

$$w' \text{ and } p' - 2\mu i\omega'$$

analogous to the corresponding series appearing in (2.7). This is now done.

From (3.8), (3.9), and (3.16) we have

$$
\begin{aligned}
2i\omega' &= e^{\sigma z}(\sigma\overline{w} + D^*\overline{w}) \\
&= \sigma U e^{\sigma z} \sum_{n=-\infty}^{\infty} (B_n - B_{n+1}) K_n(\sigma r) e^{in\theta}
\end{aligned} \tag{5.1}
$$

Then, using (3.19), we find

$$
\begin{aligned}
p' - 2\mu i\omega' &= C_0 + \rho U \sum_{n=1}^{\infty} C_n X^n r^{-n} \cos n\theta \\
&\quad - \rho \frac{U^2}{2} e^{\sigma z} \sum_{n=-\infty}^{\infty} (B_n - B_{n+1}) K_n(\sigma r) e^{in\theta}.
\end{aligned} \tag{5.2}
$$

It remains to express the series

$$
e^{\sigma z} \sum_{n=-\infty}^{\infty} B_n K_n(\sigma r) e^{in\theta} \quad \text{and} \quad e^{\sigma z} \sum_{n=-\infty}^{\infty} B_{n+1} K_n(\sigma r) e^{in\theta} \tag{5.3}
$$

in the form of Fourier series. To do this we assume

$$
e^{\sigma z} \sum_{n=-\infty}^{\infty} B_m K_m(\sigma r) e^{im\theta} = \sum_{n=-\infty}^{\infty} \mathrm{X}_n(r) e^{in\theta}. \tag{5.4}
$$

Whence, using the fact that,

$$
x = r\cos\theta
$$

we have

$$
\begin{aligned}
\mathrm{X}_n(r) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sigma r\cos\theta} \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) e^{i(m-n)\theta}\, d\theta \\
&= \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{\sigma r\cos\theta + i(m-n)\theta}\, d\theta,
\end{aligned} \tag{5.5}
$$

$$
(n = 0, \pm 1, \pm 2, \cdots).
$$

Now it is a well-known result in connection with the theory of Bessel functions that[18]

$$
\frac{1}{\pi} \int_0^{\pi} e^{z\cos\theta} \cos p\theta\, d\theta = I_p(z),
$$

[18] G. N. Watson, op. cit., p. 181 (4).

where $p$ is a positive integer and $I_p(z)$ denotes the modified Bessel function of the first kind and order $p$.
Then, from (5.5) we have

$$X_n(r) = \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r), \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{5.6}$$

From (5.4) and (5.6) it follows that

$$e^{\sigma z} \sum_{n=-\infty}^{\infty} B_n K_n(\sigma r) e^{in\theta} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r) e^{in\theta}. \tag{5.7}$$

Similarly we have for the second series in (5.3)

$$e^{\sigma z} \sum_{n=-\infty}^{\infty} B_{n+1} K_n(\sigma r) e^{in\theta} = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_{m+1} K_m(\sigma r) I_{m-n}(\sigma r) e^{in\theta}. \tag{5.8}$$

We may now write the solution (3.19) of the equations (1.5) as follows:

$$\begin{cases} w' = u' + iv' = -\sum_{n=1}^{\infty} C_n X^n r^{-n} e^{in\theta} \\ \qquad\qquad + U \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r) e^{in\theta}, \\ p' - 2\mu i\omega' = C_0 + \rho \frac{U}{2} \sum_{n=1}^{\infty} C_n X^n r^{-n} (e^{in\theta} + e^{-in\theta}) \\ \qquad\qquad - \rho \frac{U^2}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (B_m - B_{m+1}) K_m(\sigma r) I_{m-n}(\sigma r) e^{in\theta}. \end{cases} \tag{5.9}$$

From these equations (5.9), and the equations (2.7) and (2.8) we have

$$\begin{cases} w_0' = -\sum_{n=1}^{\infty} C_n e^{in\theta} + U \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} B_m K_m I_{m-n} e^{in\theta}, \\ p_0' - 2\mu i\omega_0' = C_0 + \rho \frac{U}{2} \sum_{n=1}^{\infty} C_n (e^{in\theta} + e^{-in\theta}) \\ \qquad\qquad - \rho \frac{U^2}{2} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (B_m - B_{m+1}) K_m I_{m-n} e^{in\theta}, \\ w_0 = \frac{1}{\mu} \sum_{n=-\infty}^{\infty} (M_n a_{n-1} + N a_{1-n}) e^{in\theta}, \\ p_0 - 2\mu i\omega_0 = \sum_{n=-\infty}^{\infty} \frac{a_n}{X} e^{in\theta}, \end{cases} \tag{5.10}$$

where

$$
(5.11)\qquad \begin{cases} K_n = K_n(\sigma X), \quad I_n = I_n(\sigma X), \\ M_0 = -\tfrac{1}{2}\log_e \xi, \quad M_n = (\xi^{-2n} - 1)/4n, \\ \qquad\qquad (n = \pm 1, \pm 2, \cdots), \\ N = -M_1, \\ \xi = X/a. \end{cases}
$$

Substituting from (5.10) in (4.1) and equating coefficients of $e^{in\theta}$ we find, in addition to (3.18), the following equations for $a_n$, $B_n$, and $C_n$:

$$
(5.12)\qquad \begin{cases} \dfrac{1}{\mu}(M_n a_{n-1} + N a_{1-n}) = -C_n + U \sum_{m=-\infty}^{\infty} B_m K_m I_{m-n}, \\ \qquad\qquad (n = 2, 3, \cdots), \\ 0 = -C_1 + U \sum_{m=-\infty}^{\infty} B_m K_m I_{m-1}, \\ \dfrac{1}{\mu}(M_0 a_{-1} + N a_1) = U + U \sum_{m=-\infty}^{\infty} B_m K_m I_m, \\ \dfrac{1}{\mu}(M_{-n} a_{-n-1} + N a_{n+1}) = U \sum_{m=-\infty}^{\infty} B_m K_m I_{m+n}, \quad (n = 1, 2, 3, \cdots), \\ a_0 = C_0 X, \\ \dfrac{a_n}{X} = \rho \dfrac{U}{2} C_n - \rho \dfrac{U^2}{2} \sum_{m=-\infty}^{\infty} (B_m - B_{m+1}) K_m I_{m-n}, \quad (n = 1, 2, \cdots), \\ \dfrac{a_{-n}}{X} = \rho \dfrac{U}{2} C_n - \rho \dfrac{U^2}{2} \sum_{m=-\infty}^{\infty} (B_m - B_{m+1}) K_m I_{m+n}, \\ \qquad\qquad (n = 1, 2, \cdots). \end{cases}
$$

There are several methods of treating these equations together with the equations (3.18) but it was found that, on the whole, the following treatment is the most satisfactory. Making use of the equations (3.18) and the relations

$$K_{-n} = K_n, \qquad I_{-n} = I_n,$$

we have

$$\begin{aligned}\sum_{m=-\infty}^{\infty} B_m K_m I_{m-n} &= \sum_{m=1}^{\infty} B_m K_m I_{m-n} + \sum_{m=0}^{\infty} B_{-m} K_{-m} I_{-m-n} \\ &= \sum_{m=1}^{\infty} B_m K_m I_{m-n} + \sum_{m=0}^{\infty} B_{m+1} K_m I_{m+n} \\ &= \sum_{m=0}^{\infty} B_{m+1}(K_{m+1} I_{m+1-n} + K_m I_{m+n}) \\ &= \sum_{m=0}^{\infty} B_{m+1} Z_{1-n}^m, \qquad (n = 0, \pm 1, \pm 2, \cdots),\end{aligned} \tag{5.13}$$

where

$$Z_n^m = K_{m+1} I_{m+n} + K_m I_{m-n+1}, \qquad (n = 0, \pm 1, \pm 2, \cdots; m = 0, 1, 2, \cdots). \tag{5.14}$$

In particular we note that[19]

$$Z_0^m = K_{m+1} I_m + K_m I_{m+1} = 1/(\sigma X), \qquad (m = 0, 1, 2, \cdots). \tag{5.15}$$

In a similar manner we may show that

$$\sum_{m=-\infty}^{\infty} B_{m+1} K_m I_{m-n} = \sum_{m=0}^{\infty} B_{m+1} Z_{1+n}^m, \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{5.16}$$

Thus we may replace (5.12) and (3.18) by the following set of equations:

$$\begin{cases} \frac{1}{\mu}(M_n a_{n-1} + N a_{1-n}) = -C_n + U \sum_{m=0}^{\infty} B_{m+1} Z_{1-n}^m, \quad (n = 2, 3, \cdots), \\ 0 = -C_1 + U \sum_{m=0}^{\infty} B_{m+1} Z_0^m, \\ \frac{1}{\mu}(M_0 a_{-1} + N a_1) = U\left(1 + \sum_{m=0}^{\infty} B_{m+1} Z_1^m\right), \\ \frac{1}{\mu}(M_{-n} a_{-n-1} + N a_{n+1}) = U \sum_{m=0}^{\infty} B_{m+1} Z_{1+n}^m, \qquad (n = 1, 2, \cdots), \\ a_0 = C_0 X, \\ \frac{a_n}{\mu} = \sigma X\left[C_n - U \sum_{m=0}^{\infty} B_{m+1}(Z_{1-n}^m - Z_{1+n}^m)\right], \qquad (n = 1, 2, \cdots), \\ \frac{a_{-n}}{\mu} = \sigma X\left[C_n + U \sum_{m=0}^{\infty} B_{m+1}(Z_{1-n}^m - Z_{1+n}^m)\right], \qquad (n = 1, 2, \cdots). \end{cases} \tag{5.17}$$

[19] G. N. Watson, op. cit., p. 90.

Eliminating the constants $a_n$ and $C_n$ from these equations we obtain the following equations for the constants $B_n$,

$$
\begin{cases}
\displaystyle\sum_{m=0}^{\infty} B_{m+1}[Z_1^m + H_0 - G_0 - \sigma X H_0 Z_2^m] + 1 = 0, \\
\displaystyle\sum_{m=0}^{\infty} B_{m+1}\left[\left(\frac{H_{-1}}{G_{-1}} - 1\right)Z_{-1}^m + G_2 + H_2 \right. \\
\qquad \left. + \left(\frac{1}{\sigma X G_{-1}} - \sigma X H_2\right)Z_2^m - \frac{H_{-1}}{G_{-1}} Z_3^m\right] = 0, \\
\displaystyle\sum_{m=0}^{\infty} B_{m+1}\left[\left(\frac{H_{1-n}}{G_{1-n}} - 1\right)Z_{1-n}^m + \left(\sigma X G_n \frac{H_{2-n}}{G_{2-n}} + \sigma X H_n\right)Z_{2-n}^m \right. \\
\qquad + \frac{G_n}{G_{2-n}} Z_{n-1}^m + \left(\frac{1}{\sigma X G_{1-n}} - \sigma X G_n \frac{H_{2-n}}{G_{2-n}} - \sigma X H_n\right)Z_n^m \\
\qquad \left. - \frac{H_{1-n}}{G_{1-n}} Z_{n+1}^m\right] = 0, \qquad (n = 3, 4, \cdots),
\end{cases} \tag{5.18}
$$

where

$$
G_n = N + M_n, \; H_n = N - M_n, \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{5.19}
$$

The equations (5.18) are the fundamental equations of the problem. An examination of these equations shows that the numerical values of the coefficients of $B_n$, $(n = 1, 2, \cdots)$, in these equations depend only on the values of $R$ and $\xi (= X/a)$. Thus if we neglect all save a finite number of the constants $B_n$, numerical values of the retained constants may be found by solving a finite number of linear equations when $R$ and $\xi$ are given.

When a sufficient number of the constants $B_n$ have been determined in the manner described above we may, by using the equations (5.17), calculate the corresponding values of $a_n$ and $C_n$. The exact expressions for $a_n$ and $C_n$, suitable for purposes of calculation, are given in the equations below which follow readily from (5.17):

$$
\begin{cases}
a_{-1} = \mu U \displaystyle\sum_{m=0}^{\infty} B_{m+1}(2 - \sigma X Z_2^m), \\
a_1 = \mu U \sigma X \displaystyle\sum_{m=0}^{\infty} B_{m+1} Z_2^m, \\
a_{-n} = \dfrac{\mu U}{G_{1-n}} \displaystyle\sum_{m=0}^{\infty} B_{m+1}[Z_n^m - 2N\sigma X(Z_{1+n}^m - Z_{1-n}^m)], \quad (n = 2, 3, \cdots), \\
a_n = a_{-n} + 2\mu U \sigma X \displaystyle\sum_{m=0}^{\infty} B_{m+1}(Z_{1+n}^m - Z_{1-n}^m), \qquad (n = 2, 3, \cdots), \\
C_n = (a_n + a_{-n})/(\rho U X) \\
\qquad = (a_n + a_{-n})/(2\mu\sigma X), \qquad (n = 1, 2, \cdots).
\end{cases} \tag{5.20}
$$

We note that the constants $a_0$ and $C_0$ remain undetermined but are connected by the relation

$$a_0 = C_0 X.$$

Thus the pressure in the liquid is undetermined to within an additive constant, as it should be, and this constant part of the pressure is the same throughout the liquid.

6. **Calculation of the drag.** When a viscous liquid flows past a fixed solid it is evident that the solid experiences some resultant force due to the liquid in contact with its surface. In the present case let $D$ and $D'$ denote the components of such resultant force, per unit length of cylinder, in the directions $Ox$ and $Oy$ respectively. The component $D$ is called the drag per unit length of the cylinder or briefly the drag. In order to find the value of $D$ we proceed in the following manner.

Introducing the complex stress $T$ defined by

$$T = T_x + iT_y, \tag{6.1}$$

where $(T_x, T_y)$ denote the components of stress across an element of $r = a$, we have

$$D + iD' = \oint T ds, \tag{6.2}$$

where $ds$ denotes an element of arc length measured along the section of $r = a$ by the plane of motion and the integration extends once around the cylinder in the direction of $\theta$ increasing. Now

$$T = (lE_{xx} + mE_{xy}) + i(lE_{xy} + mE_{yy}), \tag{6.3}$$

where $E_{xx}$, $E_{xy}$, and $E_{yy}$ denote the three independent components of the stress tensor evaluated at the particular point of $r = a$ under consideration and $(l, m)$ are the direction cosines of the outward normal to $r = a$.

For points along the section of $r = a$ by the plane of motion we note that,

$$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds} = i(l + im). \tag{6.4}$$

Hence, using (6.3) and the explicit expressions for $E_{ss}$, $E_{sy}$, and $E_{yy}$ in terms of the pressure and the derivatives of velocity, we have

$$\begin{aligned} T &= l\left(-p + 2\mu\frac{\partial u}{\partial x}\right)_{r=a} + m\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{r=a} + il\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)_{r=a} \\ &\qquad + im\left(-p + 2\mu\frac{\partial v}{\partial y}\right)_{r=a} \\ &= -p_a(l + im) + \mu\left[2l\frac{\partial u}{\partial x} + 2im\frac{\partial v}{\partial y} + 2il\frac{\partial u}{\partial y} + 2m\frac{\partial v}{\partial x}\right. \\ &\qquad \left. + (il - m)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]_{r=a} \\ &= i^2 p_a(l + im) + \mu\left[-2l\frac{\partial v}{\partial y} - 2im\frac{\partial u}{\partial x} + 2il\frac{\partial u}{\partial y} + 2m\frac{\partial v}{\partial x}\right]_{r=a} \\ &\qquad + 2\mu i(l + im)\omega_a \\ &= ip_a\frac{dz}{ds} + \mu\left[2i\left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right)\right. \\ &\qquad \left. - 2\left(\frac{\partial v}{\partial x}\frac{dx}{ds} + \frac{\partial v}{\partial y}\frac{dy}{ds}\right)\right]_{r=a} + 2\mu\omega_a\frac{dz}{ds} \\ &= ip_a\frac{dz}{ds} + 2\mu i\frac{dw_a}{ds} + 2\mu\omega_a\frac{dz}{ds}, \end{aligned} \tag{6.5}$$

where the subscript '$a$' denotes the value of the corresponding quantity at a point on $r = a$.

Substituting from (6.5) in (6.2) we have

$$\begin{aligned} D + iD' &= \oint (ip_a dz + 2\mu i dw_a + 2\mu\omega_a dz) \\ &= i\oint (p_a - 2\mu i\omega_a)dz. \end{aligned} \tag{6.6}$$

But from (2.7) we find

$$p_a - 2\mu i\omega_a = \sum_{n=-\infty}^{\infty} a_n z^n / X^{n+1}. \tag{6.7}$$

Thus we have, finally,

$$D + iD' = i \oint a_n \frac{z^n}{X^{n+1}}\, dz \tag{6.8}$$

$$= -2\pi a_{-1},$$

so that

$$D = -2\pi a_{-1}, \qquad D' = 0. \tag{6.9}$$

When numerical values of a sufficient number of the constants $B_n$ have been found by solving a corresponding number of the equations (5.18), the values of $R$ and $\xi$ being assigned, we can find the value of $a_{-1}$ from the first equation in (5.20) and hence, using (6.9), calculate a value for $D$. However, this value for $D$ will depend on the units employed and for this reason it is better to introduce a dimensionless quantity $k_D$, called the drag coefficient, whose value depends only on the quantities $R$ and $\xi$. In terms of $k_D$ the value of the drag is given by the equation

$$D = 2\rho U^2 a k_D\,. \tag{6.10}$$

Actually, we find by direct calculation from (6.9) and the first equation in (5.20)

$$D = k\mu U, \tag{6.11}$$

where $k$ has a definite numerical value corresponding to each pair of values for $R$ and $\xi$. The corresponding value of $k_D$ is given by

$$k_D = k/R. \tag{6.12}$$

The following table shows the calculated values of $k_D$ corresponding to a rectangular array of values for $R$ and $\xi$. The numerical values of $K_n$ and $I_n$ with argument

$$\sigma X = R\xi/4$$

were taken from the tables given by Watson[20] and by Gray, Mathews and MacRobert[21] or were calculated by means of recurrence formulae from the corresponding values of $K_0$, $K_1$, $I_0$ and $I_1$. The actual calculations were carried out on an 8-figure Monroe machine which greatly facilitated the work.

[20] G. N. Watson, op. cit.

[21] A. Gray, G. B. Mathews, and T. M. MacRobert, *A Treatise on Bessel Functions* (London, 1922).

TABLE I

*Values of $k_D$*

| $\xi$ \ $R$ | 0.4 | 0.8 | 1.6 | 3.2 | 4.8 | 6.4 | 8.0 | 20 |
|---|---|---|---|---|---|---|---|---|
| 1 | 10.67[22] | 6.78[22] | 4.61[22] | 3.25[22] | 2.72[22] | 2.43[22] | 2.24[22] | 1.74[22] |
| 1.1 | | | 4.506 | 3.15 | 2.59 | | | |
| 1.2 | | | | | | | 1.95 | 1.37 [4][23] |
| 1.5 | | 6.675 | 4.37 | 2.94 | 2.36 | 2.02 | 1.80 | |
| 1.6 | | | | | | | | 1.17 [4] |
| 2.0 | 10.50 | 6.57 | 4.22 | 2.77 | 2.185 | 1.86 | 1.65 | |
| 2.4 | | | | | 2.08 | | 1.55 | 0.97 [4] |
| 3 | 10.37 | 6.384 | 3.99 | 2.53 | 1.96 | 1.64 | 1.434 [5] | |
| 4 | 10.24 | 6.22 | 3.81 | 2.36 | 1.80 | | 1.29[24] | |
| 6 | 9.99 | 5.944 | 3.55 | 2.14 | | | | |
| 8 | 9.78 | 5.73 | 3.37 | | | | | |
| 10 | 9.595 | | | | | | | |
| 12 | 9.43 | 5.424 | | | | | | |

*Explanation of Table I.*—The drag $D$ per unit length of a cylinder of radius $a$ in a viscous liquid of density $\rho$ and viscosity $\mu$ is given by $D = 2k_D\rho U^2 a$, where $U$ is the undisturbed velocity of the stream. The drag coefficient $k_D$ depends on the Reynolds number $R = 2\rho Ua/\mu$ and the artificial parameter $\xi = X/a$, where $X$ is the radius of the cylinder on whose surface we have fitted together the solutions of Stokes' and Oseen's equations.

[22] These values for $k_D$ are those obtained from the exact solution of Oseen's equations by Bairstow in a paper to which we have referred (Phil. Trans. Roy. Soc., A 223 (1923), 383–432). The values quoted here were actually read from a smooth curve through Bairstow's results.

[23] Numbers in brackets after a result correspond to the number of the constants $B_m$, to be found by solving (5.18), which were retained in obtaining the result quoted. For the majority of the cases given in the above table, it was sufficient to retain two of the constants $B_m$ in order to obtain the corresponding value of $k_D$ with the accuracy quoted and in such cases no numbers are shown in brackets.

[24] For this value and also the values in the last column, exclusive of Bairstow's result, the last figure is somewhat doubtful on account of losses in accuracy encountered in the solution of (5.18) when $\sigma X$ is large. These losses may be partially avoided by changing the unknowns in the fundamental equations (5.18). Thus we may write

$$D_m = B_m K_m, \qquad (m = 0, \pm 1, \pm 2, \cdots),$$

and from (3.18) we have

$$D_{-m} = D_{m+1} K_m / K_{m+1}, \qquad (m = 0, 1, 2, \cdots).$$

The equations (5.18) may now be used to obtain a set of equations for the new

An examination of the values for $k_D$ given in the above table shows:

(1) For a fixed value of $R$ the value of $k_D$ decreases as $\xi$ increases and is always less than Bairstow's result for the same value of $R$. This variation of $k_D$ with $\xi$ is more pronounced for large values of $R$ than for small.

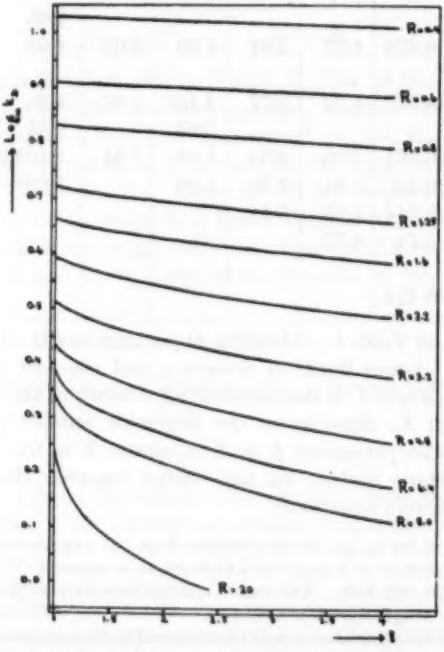

FIG. 1. CURVES SHOWING THE VARIATION OF $k_D$ WITH $\xi$ FOR SEVERAL VALUES OF THE REYNOLDS NUMBER $R$

(2) For a fixed value of $\xi$ the values of $k_D$ decrease as $R$ increases. This decrease is very rapid when $R$ is small but for larger values of $R$ it is not nearly so noticeable. We note that Bairstow's values for $k_D$ exhibit a similar dependence on $R$ and we shall see that the same is also true for the experimental values of $k_D$ provided $R$ is not too large.

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constants $D_m$, $(m = 1, 2, \cdots)$, and from the values of a sufficient number of these constants we can find the corresponding value of the drag coefficient $k_D$.

By graphic interpolation from the results in Table I, and extrapolation where the latter was justifiable, a number of additional values for $k_D$ were obtained and from these values and those in Table I the author was able to plot the curves shown in Figs. 1–2.

The curves in Fig. 1 show the variation of $\log_{10} k_D$ with $\xi$ for several values of $R$. From an examination of these curves it may be seen that, for each value[25] of $R$, a smooth curve through the results of our calculations cuts the axis $\xi = 1$ in a point corresponding to Bairstow's result. Consequently we may regard Bairstow's method as a particular

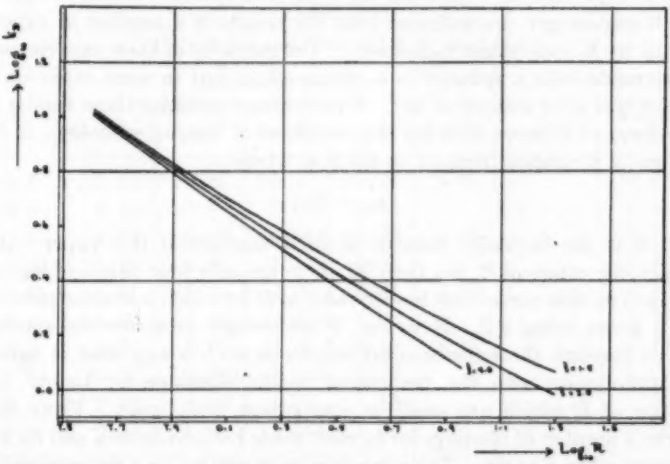

FIG. 2. CURVES SHOWING THE VARIATION OF $k_D$ WITH $R$ FOR DIFFERENT VALUES OF $\xi$

case of the method used in this paper, namely that case in which the Stokes motion is ignored altogether.

The curves in Fig. 2 show the variation of $\log_{10} k_D$ with $\log_{10} R$ for several values of $\xi$. We note that these curves are very close together when $R$ is small but diverge as $R$ increases. If we could produce these curves indefinitely in the direction of $\log_{10} R$ decreasing, it appears that

[25] The particular case $R = 1.6$ seems to be an exception. If, when plotting the curve between $\log_{10} k_D$ and $\xi$ for this case, we use a more extended scale than that shown in Fig. 1, it is found that a smooth curve through the results of our calculations cuts the axis $\xi = 1$ in a point corresponding to a value for $k_D$ which is about one per cent less than Bairstow's value. However this discrepancy may be due to small errors in graphic interpolation and extrapolation.

these curves would merge into one another. This indicates that $k_D$ is practically independent of $\xi$ when $R$ is sufficiently small, a result which also appears from the curves in Fig. 1.

It remains to compare the results of our calculations with the experimental data which is not very satisfactory for values of $R$ in the range in which we are primarily interested, viz., $0 < R \leq 20$. The experimental results which were actually used in making this comparison are those given by Wieselsberger[26] but we shall also refer to some other results given by Relf and by Eisner.

The experimental results for the drag on a cylinder, which are given by Wieselsberger, are collected from the results of a number of experiments by Wieselsberger and others. The majority of these experiments were made with a cylinder in a stream of air but in some cases oil or water was used instead of air. Wieselsberger exhibits these results in the form of a curve showing the variation of $\log_{10} c_w$ with $\log_{10} R$ for values of $R$ ranging from 0.1 to 800,000, where

$$c_w = 2k_D$$

and $R$ is the Reynolds number as used throughout this paper. Actually, for values of $R$, less than 20, there are only four values of $\log_{10} c_w$ marked on this curve—the lowest value of $R$ for which a measurement of $c_w$ is given being 4.2. However, Wieselsberger produces the smooth curve through the experimental results in such a way that it agrees asymptotically with the theoretical results obtained by Lamb[27] for values of $R$ which are small in comparison with unity. From this curve a number of readings for $c_w$ were made both by myself and by my colleague Mr. Rannie. These readings were plotted on a more extended scale than is used by Wieselsberger and a smooth curve was drawn through the corresponding points. From this curve I was able to deduce the values of $k_D$ which are given in Table II under the heading, "Observed results—Wieselsberger".

In 1914 Relf[28] made a series of experiments using wires of circular cross-section in air and found values for $k_D$ corresponding to values of $R$ in the range from 10 to 20,000. In his paper, published in 1923, Bairstow[29] compared his results for $k_D$ with the experimental values found by Relf. In order to make this comparison, Bairstow reproduces a number of Relf's values for $k_D$. These were plotted and from a smooth curve

[26] Loc. cit.
[27] Loc. cit.
[28] Loc. cit.
[29] Bairstow et al., Phil. Trans. Roy. Soc. A, 223, (1923), 383–423.

through them, were read the values given in Table II under the heading, "Observed results—Relf".

In 1930 Eisner[30] made a survey of the resistance problem for a solid in a liquid and in this communication he shows a curve similar to that given previously by Wieselsberger. In fact Eisner's curve is plotted from Wieselsberger's results together with some additional values for $c_w$. A number of readings were made from this curve but the results agree so closely with the readings from Wieselsberger's curve that it is felt unnecessary to tabulate these values here.

TABLE II

*Calculated and Observed Values of $k_D$*

| $R$ | Calculated Results | | | | | Observed Results | |
|---|---|---|---|---|---|---|---|
| | Bairstow | $\xi = 2$[31] | $\xi = 3$[31] | $\xi = 4$[31] | $\xi = 6$[31] | Relf | Wieselsberger |
| 0.4 | 10.67 | 10.50 | 10.37 | 10.24 | 9.99 | | 10.0 |
| 0.6 | 8.13 | 7.98 | 7.81 | 7.65 | | | 7.4 |
| 0.8 | 6.78 | 6.57 | 6.384 | 6.22 | 5.94 | | 6.0 |
| 1.25 | 5.26 | 4.91 | 4.71 | 4.53 | | | 4.46 |
| 1.6 | 4.61 | 4.22 | 3.99 | 3.81 | 3.55 | | 3.79 |
| 2.2 | 3.89 | 3.46 | 3.22 | 3.05 | | | 3.05 |
| 3.2 | 3.25 | 2.77 | 2.53 | 2.36 | 2.14 | | 2.43 |
| 4.8 | 2.72 | 2.185 | 1.956 | 1.80 | | | 1.94 |
| 6.4 | 2.43 | 1.86 | 1.64 | 1.49 | | | 1.66 |
| 8.0 | 2.24 | 1.65 | 1.434 | 1.29 | | | 1.49 |
| | | $\xi = 1.2$ | $\xi = 1.6$ | $\xi = 2.4$ | | | |
| 10.0 | 2.09 | | | | | 1.60 | 1.33 |
| 12.0 | 1.98 | | | | | 1.47 | 1.22 |
| 16.0 | 1.85 | | | | | 1.25 | 1.09 |
| 20.0 | 1.74 | 1.37 | 1.17 | 0.97 | | 1.11 | 1.02 |

From an examination of the values in the above table we are led to make the following conclusions:

(1) Bairstow's values for $k_D$ are in excess of the experimental values but the difference is not great for values of $R$ less than unity. As we

[30] F. Eisner, Proceedings of the Third International Congress for Applied Mechanics, I (1930), 23–41.

[31] Some of the values in these columns were obtained by interpolation from the results in Table I.

have already stated, Bairstow compared his results with those of Relf, which are evidently somewhat higher than those of Wieselsberger for values of $R$ less than 20. In this way Bairstow found that his results seemed to agree with experiment more closely than the values given in the above table would indicate.

(2) For a fixed value of $R$, the values of $k_D$, calculated for different values of $\xi$, decrease as $\xi$ increases and ultimately lie below the experimental value. Accordingly, if, for any value of $R$ in the range considered, we use a value of $\xi$ slightly larger than unity, we obtain a value

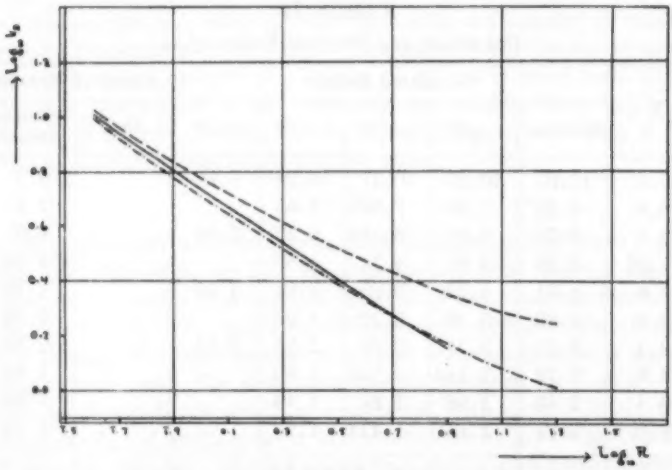

FIG. 3. ———, calculated values of $k_D$ for $\xi = 3$. — — —, Bairstow's values for $k_D$. —·—·—, experimental values of $k_D$ according to Wieselsberger.

for $k_D$ which agrees more closely with the experimental result than does the value given by Bairstow. In fact, by a proper choice of $\xi$ corresponding to each value of $R$, it is clear that we can obtain a result which agrees exactly with the experimental value of $k_D$. If we denote this value of $\xi$ by the symbol $\xi_R$ we note that $\xi_R$ decreases as $R$ increases and lies in the approximate range $2 < \xi_R < 6$ for values of $R$ less than 20 and greater than 0.4.

In Fig. 3 a number of curves have been plotted using the data in Table II. These curves substantiate the conclusions given above and

show that the values of $k_D$ for $\xi = 3$ are in good agreement with the experimental values for $R < 10$. Obviously, a smaller value of $\xi$ than 3 would improve the agreement with experiment for larger values of $R$ but the agreement would be less satisfactory for small values of $R$.

Using curves similar to those shown in Fig. 1 we can, by interpolation, determine a value for $\xi_R$ corresponding to each value of $R$ considered. In this way we find:

TABLE III

*Values of* $\xi_R$

| $R$ | $\xi_R$ |
|---|---|
| 0.4 | 6.0 |
| 0.6 | 5.7 |
| 0.8 | 5.5 |
| 1.25 | 4.4 |
| 1.6 | 4.1 |
| 2.2 | 4.0 |
| 3.2 | 3.6 |
| 4.8 | 3.0 |
| 6.4 | 2.8 |
| 8.0 | 2.7 |
| 20 | 2.2 |

These values of $\xi_R$, with the exception of the last four, are subject to rather large possible errors. This is due to the fact that, when $R$ is small the value of $k_D$ varies very slowly with $\xi$. Consequently, for such values of $R$, a small error in reading the experimental value of $k_D$ changes the value of $\xi_R$ considerably. Thus, in the worst case, viz. $R = 0.4$, it is found that the possible error in the value of $\xi_R$ is about 10 per cent. Fortunately the converse result is also true, so that, when $R$ is small a change of 5 or 10 per cent in the value of $\xi$ does not alter the corresponding value of $k_D$ appreciably.

The curve in Fig. 4 was obtained by plotting $\xi_R$ against $\log_{10} R$ and this curve shows that as $R$ increases the decrease in $\xi_R$ becomes less pronounced—in fact the curve tends to become parallel to the axis of $\log_{10} R$ when $R$ becomes large. The result that $\xi_R$ decreases as $R$ increases might have been anticipated from general considerations of the actual flow of a liquid past a cylinder seeing that we would expect

the Oseen regime to extend further in toward the cylinder when the undisturbed velocity of the stream is increased.

An important question now arises, viz., "What value of $\xi$ are we to use in determining the value of $k_D$ corresponding to a given value of $R$?" If we are merely interested in predicting a value for $k_D$ which will agree with Wieselsberger's experimental value, the answer is clearly $\xi_R$. However, if we wish to know the value of $k_D$ predicted by hydrodynamical theory alone, there appears to be no definite answer, since $\xi$

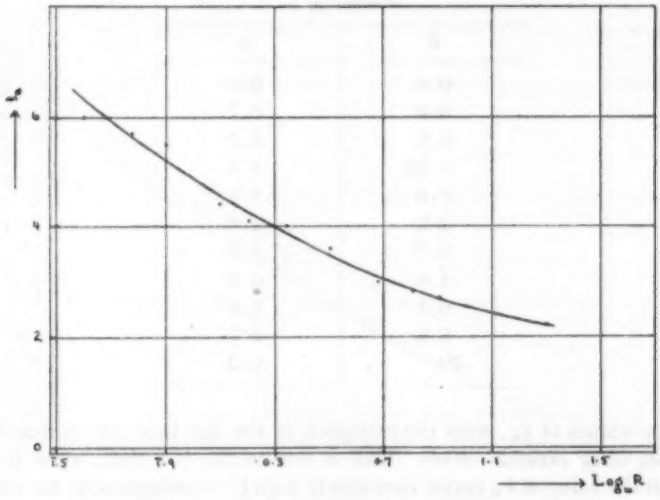


FIG. 4. CURVE SHOWING THE VARIATION OF $\xi_R$ WITH $R$

is an artificial parameter for which we may choose any real value greater than unity.

In order that our theory may give a definite answer to the above question, it is clear that we must adopt some criterion to determine a value $\xi_T$ which, for theoretical reasons, may be regarded as the proper value of $\xi$ corresponding to a given value of $R$. For example, we might agree to determine $\xi_T$ by insisting that the proper value of $\xi$, for a given value of $R$, is that value such that on the surface $r = a\xi_T$ the magnitude of the actual velocity $(u^2 + v^2)^{\frac{1}{2}}$ and the magnitude of the disturbance velocity $(u'^2 + v'^2)^{\frac{1}{2}}$ have the same average value—the average being taken with respect to the arc length measured along the section of

$r = a\xi_T$ by the plane of motion. Again, we could agree that $\xi_T$ is that value of $\xi$ for which a certain integral over the plane of motion is a minimum, viz., the integral of the magnitude of the extraneous body forces required to make our solution satisfy the exact equations (1.1).

In view of the arbitrary nature of such a theoretical criterion, and also in view of the labour involved in making the corresponding calculations, it was deemed advisable to omit the detailed consideration of such a criterion.

We may now summarize the results of our calculations for the drag in the following manner. The method given in this paper leads to values for the drag coefficient which depend on two parameters $R$ and $\xi$. The parameter $R$ is a natural one while $\xi$ is an artificial parameter whose value is arbitrary. This arbitrary feature cannot be avoided since any theoretical criterion, which we might adopt in order to determine a value for $\xi$, corresponding to a given value of $R$, is in itself arbitrary. Indeed, in this respect, our method does not differ greatly from that of Oseen, which, while apparently quite definite, may actually be regarded as a particular case of the method used here.[32] Accordingly, we should not attach more importance to Bairstow's results for $k_D$ than to the results of our calculations for a fixed value of $\xi$. Moreover, if we use a fixed value of $\xi$ in the range $1 < \xi \leq 4$, we obtain results for $k_D$ which agree more closely with the experimental values, throughout the range $0 < R \leq 20$, than do those of Bairstow.

With reference to the artificial parameter $\xi$, for which the choice of the value $\xi_R$ enables us to obtain a value for $k_D$ in agreement with the experimental value, the following remarks may be of interest. If we examine photographs of the actual flow of a liquid past a fixed cylinder, for example those given by Thom[33] and by Tietjens,[34] we see that actually the Stokes approximation is not valid for a value of $\xi$ as large as $\xi_R$. This does not invalidate our theory but simply means that when we use the value $\xi_R$ for $\xi$ we are applying our method in a case where our basic hypothesis is not justified. In other words, we cannot say that, for $r \leq a\xi_R$ the method of Stokes is a better approximation to

[32] Professor J. L. Synge has also pointed out that Oseen's method may be applied with $U$ replaced by $cU$, where $c$ is any constant having a value between zero and unity. In fact $c = 1$ gives us the Oseen solution whereas $c = 0$ corresponds to the method of Stokes.

[33] Loc. cit.

[34] L. Prandtl and O. G. Tietjens, Applied Hydro- and Aeromechanics (Engineering Societies Monographs, 1934).

the actual flow, or to the solution of the exact equations (1.1), than the method of Oseen.

7. **The determination of the stream function.** For a liquid whose motion is two-dimensional the stream function $\psi$ is a function of position such that

$$u = -\frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \psi}{\partial x}, \tag{7.1}$$

where $(u, v)$ are the components of velocity at a point $(x, y)$ in the liquid, the axes being rectangular. If we use polar coordinates $(r, \theta)$ and denote the radial and transverse components of velocity by $u_r$ and $u_\theta$ respectively, the equations corresponding to (7.1) are

$$u_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad u_\theta = \frac{\partial \psi}{\partial r}. \tag{7.2}$$

We note that, by virtue of the equation of continuity, $\psi$ will always exist for the two-dimensional flow of a liquid.

The equations (7.1), or equivalently (7.2), may be used to find $\psi$ when the velocity components are known *or* to find the velocity components when $\psi$ is known. The more usual method of treating the two-dimensional motion of a liquid is to find $\psi$ as the solution of a partial differential equation, found by eliminating the pressure from the equations of motion, and then to use the equations (7.1) in order to calculate $u$ and $v$. However, in the present case, we have already found expressions for $u$ and $v$ so that we shall use (7.1) to find $\psi$. Actually it is better to use the known expressions for $u$ and $v$ to calculate $u_r$ and $u_\theta$, and then to use the equations (7.2) in finding $\psi$.

Since our method requires us to use two different solutions for the velocity components we must actually find two stream functions $\psi$ and $\psi'$. The function $\psi$ will hereafter denote the stream function corresponding to the solution of Stokes' equations, while $\psi'$ will denote the stream function corresponding to the solution of Oseen's equations.

To determine $\psi$ we use the equation (7.2) where $(u_r, u_\theta)$ are calculated from the expressions for $u$ and $v$ given by (2.7). By resolution of velocities we have

$$u_r = u\cos\theta + v\sin\theta, \qquad u_\theta = v\cos\theta - u\sin\theta. \tag{7.3}$$

Hence, using (2.7) and (7.2), we find the following equations for $\psi$:

$$(7.4)\qquad \begin{cases} -\dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = \displaystyle\sum_{n=-\infty}^{\infty} f_n(r)\cos(n-1)\theta, \\ \dfrac{\partial \psi}{\partial r} = \displaystyle\sum_{n=-\infty}^{\infty} f_n(r)\sin(n-1)\theta, \end{cases}$$

where $f_n(r)$ are given by (2.8). Of course the two equations in (7.4) are not independent and are only compatible on account of the fact that our solution (2.7) of the equations (1.3) actually satisfies the equation of continuity.

From (2.8) we see that

$$f_1(r) = 0$$

and hence, integrating the first equation in (7.4) subject to the boundary condition that $\psi = 0$ when $\theta = 0$, we find

$$(7.5)\qquad \psi = -\sum_{n=-\infty}^{\infty}{}' \frac{r f_n(r)}{n-1}\sin(n-1)\theta,$$

where the accent on $\sum$ means that the term corresponding to $n = 1$ is omitted in the summation. By actual differentiation it is easily verified that $\psi$, as given by (7.5), also satisfies the second equation in (7.4). Thus (7.5) gives the proper expression for $\psi$ corresponding to all points in the region $a \leq r \leq X$.

To determine $\psi'$ we must write $\psi'$ in place of $\psi$ in the equations (7.2) and then use (1.4) together with (7.3) in order to find $u_r$, $u_\theta$ in terms of known expressions. In this way we are led to the equations

$$(7.6)\qquad \begin{cases} -\dfrac{1}{r}\dfrac{\partial \psi'}{\partial \theta} = U\cos\theta + u'\cos\theta + v'\sin\theta, \\ \dfrac{\partial \psi'}{\partial r} = -U\sin\theta - u'\sin\theta + v'\cos\theta, \end{cases}$$

where $(u', v')$ are given by the solution (3.19) of Oseen's equations (1.5) *or* equivalently by (5.9). In carrying out the evaluation of $\psi'$ it is more convenient to use the expressions given by (5.9) for $(u', v')$. Thus, substituting from (5.9) in (7.6), we have

$$
(7.7)\quad \begin{cases} -\dfrac{1}{r}\dfrac{\partial \psi'}{\partial \theta} = U\cos\theta - \displaystyle\sum_{n=1}^{\infty} C_n \frac{X^n}{r^n}\cos(n-1)\theta \\ \qquad\qquad + U\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r)\cos(n-1)\theta, \\ \dfrac{\partial \psi'}{\partial r} = -U\sin\theta - \displaystyle\sum_{n=1}^{\infty} C_n \frac{X^n}{r^n}\sin(n-1)\theta \\ \qquad\qquad + U\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r)\sin(n-1)\theta. \end{cases}
$$

If we introduce the functions $Z_n^m(\sigma r)$ defined by

$$
(7.8)\quad Z_n^m(\sigma r) = K_{m+1}(\sigma r) I_{m+n}(\sigma r) + K_m(\sigma r) I_{m-n+1}(\sigma r), \qquad (n = 0, \pm 1, \pm 2, \cdots; m = 0, 1, 2, \cdots),
$$

we find, as in (5.13)

$$
(7.9)\quad \sum_{m=-\infty}^{\infty} B_m K_m(\sigma r) I_{m-n}(\sigma r) = \sum_{m=0}^{\infty} B_{m+1} Z_{1-n}^m(\sigma r), \qquad (n = 0, \pm 1, \pm 2, \cdots).
$$

We may now write the equations (7.7) in the following form:

$$
(7.10)\quad \begin{cases} -\dfrac{1}{r}\dfrac{\partial \psi'}{\partial \theta} = U\cos\theta - \displaystyle\sum_{n=1}^{\infty} C_n \frac{X^n}{r^n}\cos(n-1)\theta \\ \qquad\qquad + U\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} B_{m+1} Z_{1-n}^m(\sigma r)\cos(n-1)\theta, \\ \dfrac{\partial \psi'}{\partial r} = -U\sin\theta - \displaystyle\sum_{n=1}^{\infty} C_n \frac{X^n}{r^n}\sin(n-1)\theta \\ \qquad\qquad + U\displaystyle\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} B_{m+1} Z_{1-n}^m(\sigma r)\sin(n-1)\theta. \end{cases}
$$

If we pick out the term, on the right-hand side of the first equation in (7.10), which does not involve $\theta$, we find the expression

$$
-C_1\frac{X}{r} + U\sum_{m=0}^{\infty} B_{m+1} Z_0^m(\sigma r).
$$

Using (5.15) and the equation on the second line in (5.17) we see that this expression vanishes. Using this fact and integrating the first

equation in (7.10), subject to the boundary condition $\psi' = 0$ when $\theta = 0$, we find

$$\psi' = -Ur\sin\theta + \sum_{n=2}^{\infty} C_n \frac{X_n}{r^{n-1}} \frac{\sin(n-1)\theta}{n-1} - Ur \sum_{n=-\infty}^{\infty}{}' \sum_{m=0}^{\infty} B_{m+1} Z^m_{1-n}(\sigma r) \frac{\sin(n-1)\theta}{n-1}, \tag{7.11}$$

where, as in (7.5), the accent on $\sum$ means that the term corresponding to $n = 1$ is omitted in the summation.

Again, it is easily verified that the expression for $\psi'$, given by (7.11), actually satisfies the second equation in (7.10). In carrying out the differentiation involved in checking this fact we find the result

$$\frac{d}{dt}\left\{\frac{t}{n}[Z^m_n(t) + Z^m_{-n}(t)]\right\} = Z^m_n(t) - Z^m_{-n}(t), \qquad (n = 1, 2, \cdots; m = 0, 1, 2, \cdots). \tag{7.12}$$

By integration we have, from (7.12)

$$\int [Z^m_n(t) - Z^m_{-n}(t)]\,dt = \frac{t}{n}[Z^m_n(t) + Z^m_{-n}(t)] + \text{constant}, \qquad (n = 1, 2, \cdots; m = 0, 1, 2, \cdots). \tag{7.13}$$

This is an interesting but well-known result in connection with Bessel functions. It is merely a particular case of a general formula due to Lommel.[35]

Collecting our results for the stream function we have

$$\begin{cases} \dfrac{\psi}{Ua} = -\displaystyle\sum_{n=-\infty}^{\infty}{}' \frac{r}{a}\frac{f_n(r)}{U} \frac{\sin(n-1)\theta}{n-1}, \\ \dfrac{\psi'}{Ua} = -\dfrac{r}{a}\sin\theta + \displaystyle\sum_{n=2}^{\infty} \frac{C_n}{U}\xi^n \left(\frac{r}{a}\right)^{-n} \frac{\sin(n-1)\theta}{n-1} \\ \qquad\qquad - \dfrac{r}{a}\displaystyle\sum_{n=-\infty}^{\infty}{}' \sum_{m=0}^{\infty} B_{m+1} Z^m_{1-n}(\sigma r) \frac{\sin(n-1)\theta}{n-1}, \end{cases} \tag{7.14}$$

[35] See G. N. Watson, op. cit. p. 136.

where

$$(7.15)\quad \begin{cases} \dfrac{f_0(r)}{U} = \dfrac{a_1}{\mu U}\dfrac{\left(\dfrac{r}{a}\right)^2 - 1}{4\xi^2} - \dfrac{a_{-1}}{\mu U}\tfrac{1}{2}\log_e\dfrac{r}{a}, \\ \dfrac{f_n(r)}{U} = \dfrac{a_{1-n}}{\mu U}\dfrac{\left(\dfrac{r}{a}\right)^2 - 1}{4\left(\dfrac{r}{a}\right)^n\xi^{2-n}} - \dfrac{a_{n-1}}{\mu U}\dfrac{\left(\dfrac{r}{a}\right)^{2n} - 1}{4n\left(\dfrac{r}{a}\right)^n\xi^n}, \quad (n = \pm 1, \pm 2, \cdots), \\ \xi = X/a, \end{cases}$$

and $Z_n^m(\sigma r)$ are given by (7.8). In connection with the above expressions for the stream function we should remark that they have been deduced on the assumption that the several series converge and admit the requisite term by term processes.

We note that $\psi/Ua$ is a dimensionless quantity whose numerical value at a point, for which $r/a$ is less than $\xi$, depends only on the parameters $R$ and $\xi$, in addition to the values of $r/a$ and $\theta$ for the point in question. The same remark is applicable to $\psi'/Ua$ when $r/a$ is greater than $\xi$. The truth of this statement is evident from the fact that the determination of any finite number of the constants $B_n$, $(n = 1, 2, \cdots)$, from (5.18) depends only on $R$ and $\xi$, and from these values of $B_n$ we may find the values of $a_n/\mu U$ and $C_n/U$ by means of (5.20).

When the numerical values of $\psi/Ua$ are known for a sufficient number of points in the region $a \leq r \leq X$, it is an easy matter to plot the stream-lines in this region. We have merely to determine, by interpolation, the locus of points for which $\psi/Ua$ (or $\psi$) is constant. Similarly, when the values of $\psi'/Ua$ are known for a sufficient number of points for which $r$ exceeds $X$, we can plot the stream lines in the region outside $r = X$. Now, since the solutions of Stokes' and Oseen's equations have been fitted together on $r = X$ by making velocity and stress continuous across this surface, it follows that the partial derivatives of $\psi$, of the first and second orders, are continuous across this surface. Moreover the equations (7.4) and (7.10) determine $\psi$ and $\psi'$ to within an additive constant. Then, since at $r = X$, $\theta = 0$, we have

$$\psi = \psi' = 0,$$

we see that the stream-lines must be a set of curves which are not only continuous but are also such that the slope and the curvature of each stream-line are continuous across $r = X$.

**8. Conclusion.** The equations of Stokes and Oseen have been used to determine the distribution of velocity and pressure in a liquid, of viscosity $\mu$ and density $\rho$, flowing past a fixed cylinder of radius $a$, the velocity at infinity being $(U, 0)$ referred to the special axes which we have employed. In making such a determination we have assumed that the solution of Stokes' equations is valid inside a fictitious cylinder of radius $X$, concentric with the fixed cylinder, and that, within this region, the Stokes solution constitutes a better approximation to the solution of the exact equations (1.1) than the solution of Oseen's equations. At points whose distance $r$ from the axis of the fixed cylinder exceeds $X$, we have used the solution of Oseen's equations, this solution being joined to the corresponding solution of Stokes' equations on the surface $r = X$ by using the continuity of velocity and stress.

The expressions obtained for velocity and pressure are used to find a formula for the drag $D$ per unit length of the fixed cylinder. It is found that $D$ depends on two parameters $R(= 2\rho Ua/\mu)$ which is a natural parameter, and $\xi(= X/a)$, which is an artificial parameter introduced by our method of determining the motion.

The numerical results for the drag are given in terms of a dimensionless drag coefficient $k_D$, such that

$$D = 2k_D\rho U^2 a.$$

The dependence of $k_D$ on the parameters $R$ and $\xi$ is illustrated by the values in Table I and the curves plotted in Figs. 1–2. These results are compared with the experimental values of $k_D$ and also with other theoretical results in Table II and Fig. 3. This comparison shows that, provided $\xi$ is not too large, our values for $k_D$ agree more closely with experiment than the values found from an exact solution of Oseen's equations. In Table III are shown a number of values for the quantity $\xi_R$ which is that value of $\xi$, corresponding to a given value of $R$, such that the value of $k_D$, calculated by our method, agrees with the experimental value given by Wieselsberger. It is pointed out that observations indicate that $\xi$ should not be taken to have a value as large as $\xi_R$ so that actually our method would lead to results for $k_D$ which lie between the experimental values and the theoretical results of Bairstow.

The numerical computation of $k_D$, when $R$ and $\xi$ are known, depends upon the solution of the equations (5.18) which involve an infinite number of the unknown constants $B_n$. In solving these equations for $B_n$ it was assumed that we could neglect all save the first few of these constants and solve for the others from a corresponding number of the

equations (5.18). The results of our calculations have amply justified the validity of this assumption.

Finally, expressions for the stream function are given corresponding to the solutions inside and outside the cylinder $r = X$. From these expressions it is shown that we may plot the stream-lines for the motion corresponding to our solution, when $R$ and $\xi$ are given.

The present paper is an account of investigations undertaken by the author in preparing a thesis for the degree of Doctor of Philosophy at the University of Toronto. The problem was suggested by Professor J. L. Synge, Sc.D., Head of the Department of Applied Mathematics in the University of Toronto, who directed this research and to whom I wish to take this opportunity of expressing my most grateful thanks.

My thanks are also due to my colleague, Mr. W. D. Rannie who made a large number of readings for the experimental values of $k_D$ from Wieselsberger's curve in order to check the readings which I had previously made.

University of Toronto.

# ON PERIODIC ORBITS IN THE EQUATORIAL PLANE OF A MAGNETIC DIPOLE

By C. Graef* and S. Kusaka

The periodic circular orbit in the equatorial plane of a magnetic dipole is well known. Störmer[1] discovered that in the meridian plane there is an infinity of periodic motions, among which may be distinguished the pair of periodic orbits which have the property of intersecting the equator at right angles (principal periodic orbits). Lemaitre and Vallarta[2] then showed that these do not exist for all values of $2\gamma_1$, the axial component of the angular momentum of the particle at infinity; but only in the range (in appropriate units) $0.783 < \gamma_1 < 1$. The lower limit was later refined by Lemaitre[3] to 0.78856. The value $\gamma_1 = 1$ corresponds to the periodic circular orbit in the equatorial plane, while for the value 0.78856 the two principal periodic orbits collapse together and then vanish. For values of $\gamma_1 > 1$ the outer principal orbit vanishes, but the inner orbit continues to exist. It seems that the outer orbit has collapsed into the equator, but from general theorems of Poincaré it is known that periodic orbits can only vanish in pairs, and therefore that companion orbit to the inner orbit should still exist. The present paper presents an attempt to calculate the periodic orbits in the equatorial plane. It will be shown that there is a bounded region of motion only for $\gamma_1 \geqq 1$ and that in the bounded region closed periodic orbits exist for a denumerable set of values of $\gamma_1 > 1$, while for all other values of $\gamma_1 > 1$ the motion is conditioned periodic. For any value of $\gamma_1$ the motion is unbounded in the unbounded region of motion. Several properties of these equatorial orbits will be established and the asymptotic motions will be investigated.

**The Equations of Motion and the General Integral.** The equations of motion, in polar coördinates, of a charged particle in the equatorial plane of a magnetic dipole are

* Fellow of the John Simon Guggenheim Memorial Foundation.

[1] C. Störmer, Zeits. f. Astrophys., *1*, 237 (1930).

[2] G. Lemaitre and M. S. Vallarta, Phys. Rev., *43*, 87 (1933).

[3] G. Lemaitre, Ann. de la Soc. Sci. de Bruxelles, *A 54*, 194 (1934).

$$\frac{m}{qM}\left\{\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\varphi}{dt}\right)^2\right\} = -\frac{1}{\rho}\frac{d\varphi}{dt} \tag{1}$$

$$\frac{m}{qM}\frac{1}{\rho}\frac{d}{dt}\left(\rho^2\frac{d\varphi}{dt}\right) = \frac{1}{\rho^2}\frac{d\rho}{dt} \tag{2}$$

where $M$ is the moment of the dipole, $q$ the charge, and $m$ the relativistic mass of the particle. From the conservation of kinetic energy we have

$$\left(\frac{d\rho}{dt}\right)^2 + \rho^2\left(\frac{d\varphi}{dt}\right)^2 = v^2 \tag{3}$$

It is convenient to use the normalized coördinates introduced by Störmer

$$\rho = c_1 r \tag{4}$$

$$v\,dt = c_1\,ds \tag{5}$$

where

$$c_1 = \sqrt{\frac{M|q|}{mv}} \tag{6}$$

The physical significance of $c_1$ is that it is the radius of the periodic circular orbit. In these units equations (1), (2), and (3) become

$$\frac{d^2r}{ds^2} - r\frac{d\varphi}{ds} = -\frac{1}{r^2}\frac{d\varphi}{ds} \tag{7}$$

$$\frac{1}{r}\frac{d}{ds}\left(r^2\frac{d\varphi}{ds}\right) = \frac{1}{r^2}\frac{dr}{ds} \tag{8}$$

$$\left(\frac{dr}{ds}\right)^2 + r^2\left(\frac{d\varphi}{ds}\right)^2 = 1 \tag{9}$$

Equation (8) can be integrated immediately to give

$$r^2\frac{d\varphi}{ds} + \frac{1}{r} = 2\gamma_1 \tag{10}$$

where $2\gamma_1$ is a constant which can have all values from $-\infty$ to $+\infty$. Its meaning, apparent from (10), is the moment of momentum of the particle in the equatorial plane, at infinity. From equations (9) and (10) we get

$$\left(\frac{dr}{ds}\right)^2 = 1 - \frac{4\gamma_1^2}{r^2} + \frac{4\gamma_1}{r^3} - \frac{1}{r^4} \tag{11}$$

The differential equation of the trajectory obtained by eliminating $s$ between (10) and (11) is

$$\left(\frac{dr}{d\varphi}\right)^2\left(2\gamma_1 - \frac{1}{r}\right)^2 = r^4 - 4\gamma_1 r^2 + 4\gamma_1 r - 1 \tag{12}$$

and this gives the integral

$$\varphi = \pm\int \frac{\left(2\gamma_1 - \frac{1}{r}\right)dr}{\sqrt{r^4 - 4\gamma_1^2 r^2 + 4\gamma_1 r - 1}} + C \tag{13}$$

The possibility of two signs shows that the trajectory is symmetric about a certain radial line. In the sequel we restrict our attention to one of these symmetrical branches and shall only keep the positive sign. The integral can be simplified by introducing a new variable $y$ defined by

$$y^2 = 1 + \frac{2\gamma_1}{r} - \frac{1}{r^2} \tag{14}$$

Equation (13) then becomes

$$\varphi = \sin^{-1}\frac{y}{\sqrt{2}} \mp \frac{\gamma_1}{\sqrt{2}}\frac{1}{\sqrt{1+\gamma_1^2}}I + c \tag{15}$$

where

$$I = \int \frac{dy}{\sqrt{\left(1 - \frac{y^2}{2}\right)\left(1 - \frac{y^2}{1+\gamma_1^2}\right)}} \tag{16}$$

and the choice of the sign in (15) is determined by the choice in

$$r = \frac{1}{\gamma_1^2 \pm \sqrt{\gamma_1^2 + 1 - y^2}} \tag{17}$$

which is obtained from (14).

The intrinsic equation of motion has a particularly simple form. Since the force always acts perpendicular to the direction of motion,

$$m\kappa v^2 = v\frac{|q|M}{\rho^3} \tag{18}$$

where $\kappa$ is the curvature. In normalized coördinates this becomes

$$\kappa_1 = \frac{1}{r^3} \tag{19}$$

Now from Euler's relation

$$\frac{1}{\kappa_1} = \pm r \frac{dr}{dp} \tag{20}$$

where $p$ is the perpendicular from the pole to the tangent to the trajectory, and the positive (negative) sign is taken when the pole and the center of curvature lie on the same (opposite) side of the tangent. Thus we get

$$\pm \frac{dr}{dp} = r^2 \tag{21}$$

which gives the integral

$$\pm p + \frac{1}{r} = c \tag{22}$$

In order to relate this integral to (15) and (17) we note from (17) that the minimum value of $r$ is $\dfrac{1}{\gamma_1 + \sqrt{1+\gamma_1^2}}$ and at this point $r = p$ so we have $c = 2\gamma_1$ and (22) becomes

$$\pm p + \frac{1}{r} = 2\gamma_1 \tag{23}$$

It is interesting to note that Weyr has investigated plane curves with the property such that the passage of a unit current along the curve of given length joining two given points will produce at a third point in the plane of the curve a maximum magnetic intensity. He found that these curves have the property that the radius of curvature is proportional to the cube of the radius vector. Loria[4] has called these curves the electromagnetic curves of Emil Weyr. We have seen that the trajectories of the charged particles in the equatorial plane have this property so that they are Weyr curves.

First case $\gamma_1 < 1$

To express $\varphi$ as an elliptic integral we must consider the cases $\gamma_1 < 1$, $\gamma_1 = 1$, $\gamma_1 > 1$ separately. For $\gamma_1 < 1$ we put

$$\frac{y}{\sqrt{1+\gamma_1^2}} = \sin \psi \tag{24}$$

then

$$I = \sqrt{1+\gamma_1^2} F(\alpha, \psi) \tag{25}$$

[4] G. Loria, Spezielle Algebraische und Transzendente Ebene Kurven II, p. 220.

where $F(\alpha, \psi)$ denotes the elliptic integral of the first kind and

$$\sin \alpha = k = \sqrt{\frac{1 + \gamma_1^2}{2}} \tag{26}$$

Hence (15) and (17) give

$$\varphi = \sin (k \sin \psi) \mp \frac{\gamma_1}{\sqrt{2}} F(\alpha, \psi) + c \tag{27}$$

and

$$r = \frac{1}{\gamma_1 \pm \sqrt{1 + \gamma_1^2} \cos \psi} \tag{28}$$

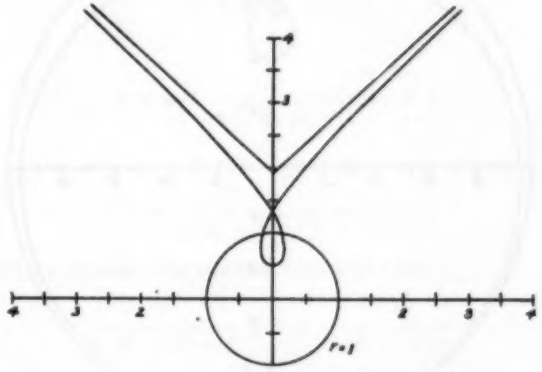

FIG. 1. AN UNBOUNDED ORBIT $\left(\gamma_1 = \frac{1}{\sqrt{2}}\right)$

In this case it is immaterial which set of signs is chosen; we shall take the lower set. Equations (27) and (28) are the parametric equations of the trajectory.

From (28) we see that $r$ takes on all values greater than $\dfrac{1}{\gamma_1 + \sqrt{1 + \gamma_1^2}}$ so the trajectories are unbounded. These trajectories have asymptotes whose directions are obtained by putting

$$\psi = \cos^{-1} \frac{\gamma_1}{\sqrt{1 + \gamma_1^2}} \text{ and } \psi = 2\pi - \cos^{-1} \frac{\gamma_1}{\sqrt{1 + \gamma_1^2}} \tag{29}$$

in (27), and whose distance from the pole is $2\gamma_1$, obtained by putting $r = \infty$ in (23). Figure 1 shows the form of the trajectory for $\gamma_1 = 1/\sqrt{2}$.

Second case $\gamma_1 = 1$

In this case

$$I = \int \frac{dy}{1 - \frac{y^2}{2}} = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2} + y}{\sqrt{2} - y} \tag{30}$$

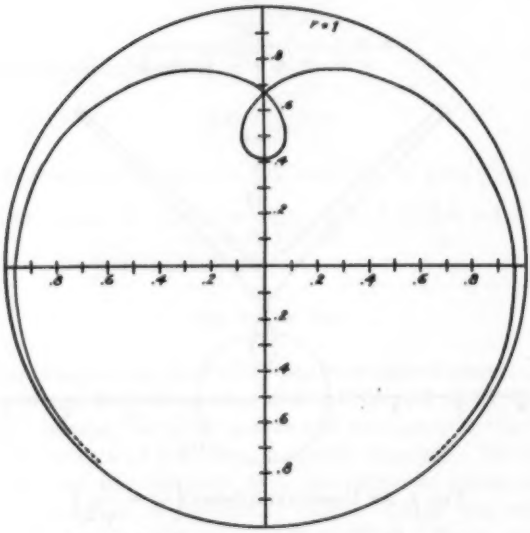


FIG. 2. DOUBLY ASYMPTOTIC ORBIT ($\gamma_1 = 1$)

Hence

$$\varphi = \sin^{-1} \frac{y}{\sqrt{2}} \mp \frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2} + y}{\sqrt{2} - y} + c \tag{31}$$

and

$$r = \frac{1}{1 \pm \sqrt{2 - y^2}} \tag{32}$$

The two sets of signs give trajectories outside and inside the unit circle. Since from (31) and (32) $\varphi$ becomes infinite as $r$ approaches unity, and since the speed is constant, the particle takes infinite time to reach the unit circle. The bounded orbit is doubly asymptotic to the

unit circle, as shown in Figure 2, and the unbounded orbit (external to the circle $r = 1$) is asymptotic to the unit circle and to a straight line.

Third case $\gamma_1 > 1$

Here we put

$$\frac{y}{\sqrt{2}} = \sin \psi \tag{33}$$

then

$$I = \sqrt{2}\, F(\alpha, \psi) \tag{34}$$

where

$$\sin \alpha = \sqrt{\frac{2}{1 + \gamma_1^2}} \tag{35}$$

Hence

$$\varphi = \psi \mp \frac{\gamma_1}{\sqrt{1 + \gamma_1^2}}\, F(\alpha, \psi) + c \tag{36}$$

and

$$r = \frac{1}{\gamma_1 \pm \sqrt{\gamma_1^2 + 1 - 2 \sin^2 \psi}} \tag{37}$$

From (37) we see that the allowed values of $r$ are

$$\frac{1}{\gamma_1 + \sqrt{\gamma_1^2 + 1}} \leq r \leq \frac{1}{\gamma_1 + \sqrt{\gamma_1^2 - 1}}$$

and

$$r \geq \frac{1}{\gamma_1 - \sqrt{\gamma_1^2 - 1}}$$

Thus there are bounded and unbounded orbits given by the two sets of signs.

The latter consist of periodic and conditioned periodic orbits and the values of $\gamma_1$ which give the periodic orbits form a set of measure zero. To see this, we note that an orbit will be periodic if the periods of $r$ and $\varphi$ are commensurable. From (36) and (37) we see that this condition is expressed by the equation

$$\frac{\dfrac{\gamma_1}{\sqrt{1 + \gamma_1^2}}\, F(\alpha, \pi) - \pi}{\pi} = 2\frac{l}{l} \tag{38}$$

where $t$ and $l$ are whole numbers. If they have no common divisor, $l$ and $t$ refer to the number of loops and the number of turns respectively in a complete period. If they have a common divisor $d$, then $t$ and $l$ are the total numbers of turns and loops respectively of a set of $d$ identical orbits. There is a one to one correspondence between the values of $t/l$ and $\gamma_1$ since $t/l$ is a monotonic function of $\gamma_1$. This can

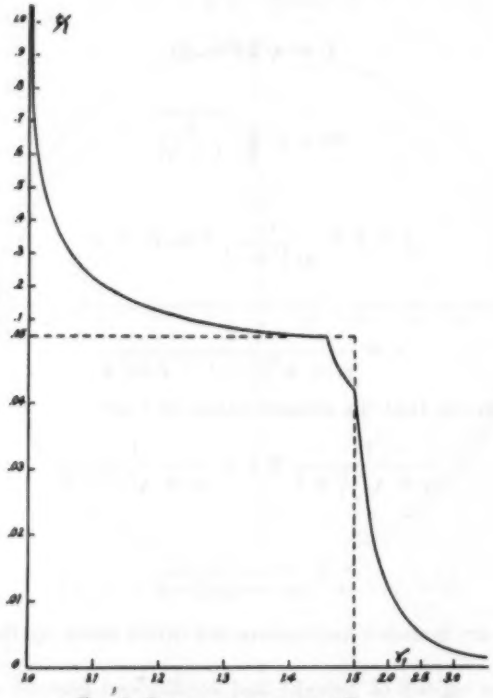

FIG. 3. RELATION BETWEEN $t/l$ AND $\gamma_1$

be seen from the graph of relation (38) in Figure 3 or analytically as follows: (38) can be written in the form

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{1+\frac{1}{\gamma_1^2}\cos 2\psi}} - 1 = \frac{2t}{l} \tag{39}$$

This is the same as

$$\frac{2}{\pi}\left[\int_0^{\frac{\pi}{4}} \frac{d\psi}{\sqrt{1+\frac{1}{\gamma_1^3}\cos 2\psi}} + \int_0^{\frac{\pi}{4}} \frac{d\psi}{\sqrt{1-\frac{1}{\gamma_1^3}\cos 2\psi}}\right] - 1 = \frac{2t}{l} \tag{40}$$

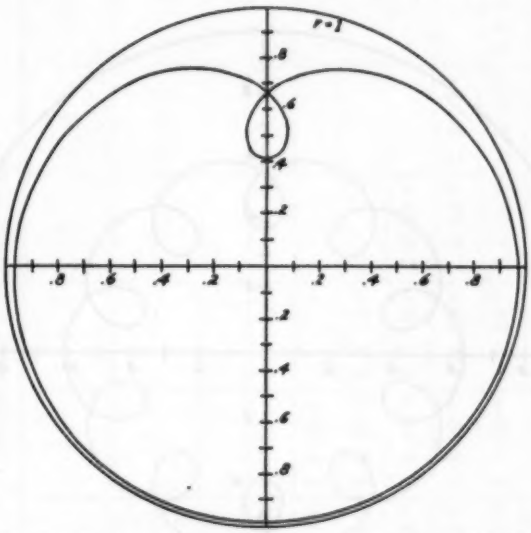


FIG. 4. PERIODIC ORBIT WITH $t = 1$, $l = 1$ ($\gamma_1 = 1.000025$)

Hence

$$\frac{d}{d\gamma_1}\left(\frac{t}{l}\right) = \frac{1}{\pi}\left[\int_0^{\frac{\pi}{4}} \frac{\frac{1}{\gamma_1^3}\cos 2\psi\, d\psi}{\left(1+\frac{1}{\gamma_1^3}\cos 2\psi\right)^{\frac{1}{2}}} - \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\gamma_1^3}\cos 2\psi\, d\psi}{\left(1-\frac{1}{\gamma_1^3}\cos 2\psi\right)^{\frac{1}{2}}}\right] \tag{41}$$

Now the integrand of the first integral is never greater than the integrand of the first so that $d(t/l)/d\gamma_1$ is never positive. Thus $t/l$ is a monotonic decreasing function of $\gamma_1$. Further, for any positive rational number $t/l$ there is a $\gamma_1$ satisfying (38) so that in the range

$$\gamma_1' - \epsilon < \gamma_1 < \gamma_1' + \epsilon \qquad (\gamma_1 > 1)$$

there are always an infinite number of values of $\gamma_1$ for which the orbits are periodic no matter how small $\epsilon$ may be. Therefore the values of $\gamma_1$ corresponding to the periodic orbits form a denumerable set. Figures 4 and 5 illustrate the orbits with 1 loop and 1 turn, 10 loops and 1 turn, respectively. Table 1 gives the values of $\gamma_1$ of periodic orbits for a few values of $l/t$.

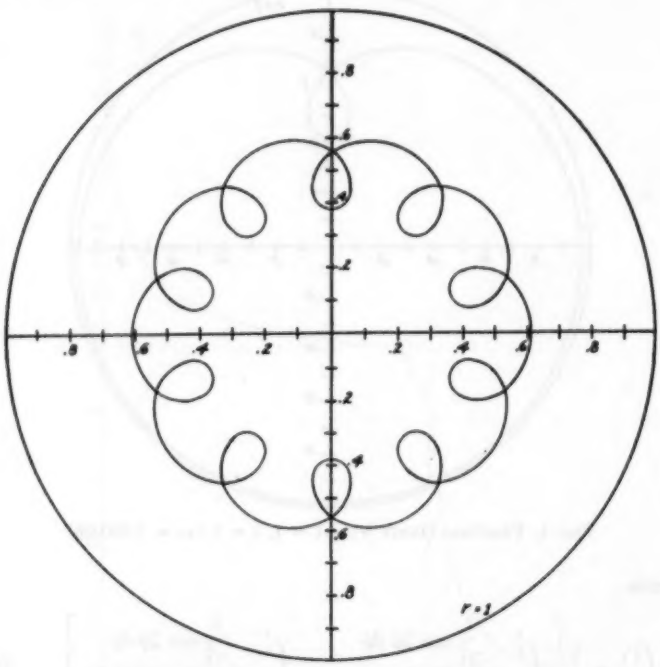


FIG. 5. PERIODIC ORBIT WITH $t = 1$, $l = 10$ ($\gamma_1 = 1.1179$)

We can also show that the doubly asymptotic orbit for $\gamma_1 = 1$ is the limiting case of the periodic orbit. For in the interval 1 to $1 + \epsilon$ we can choose at least one value of $\gamma_1$ which satisfies (38) for a given value of $l$. Now if $\epsilon$ is made very small, $t$ must become very large, and in the limit as $\epsilon$ goes to zero, $t$ becomes infinite. The result is that we have $l$ identical doubly asymptotic orbits.

TABLE I

*Values of $\delta = \gamma_1 - 1$ Corresponding to Periodic Orbits*

| $t$ \ $l$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | $2.5\cdot10^{-5}$ | | | | | | | | | |
| 2 | $3.6\cdot10^{-9}$ | $2.5\cdot10^{-5}$ | | | | | | | | |
| 3 | $5.0\cdot10^{-13}$ | $3.1\cdot10^{-7}$ | $2.5\cdot10^{-5}$ | | | | | | | |
| 4 | $6.9\cdot10^{-17}$ | $3.6\cdot10^{-9}$ | $1.3\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | | | | | | |
| 5 | $9.5\cdot10^{-21}$ | $4.2\cdot10^{-11}$ | $7.0\cdot10^{-8}$ | $2.8\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | | | | | |
| 6 | $1.3\cdot10^{-24}$ | $5.0\cdot10^{-13}$ | $3.6\cdot10^{-9}$ | $3.1\cdot10^{-7}$ | $4.4\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | | | | |
| 7 | $1.8\cdot10^{-28}$ | $5.9\cdot10^{-15}$ | $1.8\cdot10^{-10}$ | $3.3\cdot10^{-8}$ | $7.4\cdot10^{-7}$ | $5.9\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | | | |
| 8 | $2.5\cdot10^{-32}$ | $6.9\cdot10^{-17}$ | $9.5\cdot10^{-12}$ | $3.6\cdot10^{-9}$ | $1.3\cdot10^{-7}$ | $1.3\cdot10^{-6}$ | $7.3\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | | |
| 9 | $3.5\cdot10^{-36}$ | $8.1\cdot10^{-19}$ | $5.0\cdot10^{-13}$ | $3.9\cdot10^{-10}$ | $2.1\cdot10^{-8}$ | $3.1\cdot10^{-7}$ | $2.1\cdot10^{-6}$ | $8.6\cdot10^{-6}$ | $2.5\cdot10^{-5}$ | |
| 10 | $4.8\cdot10^{-40}$ | $9.5\cdot10^{-21}$ | $2.6\cdot10^{-14}$ | $4.2\cdot10^{-11}$ | $3.6\cdot10^{-9}$ | $7.0\cdot10^{-8}$ | $5.8\cdot10^{-7}$ | $2.8\cdot10^{-6}$ | $9.7\cdot10^{-6}$ | $2.5\cdot10^{-5}$ |

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# FINDING COMPLEX ROOTS OF ALGEBRAIC EQUATIONS

By Frank L. Hitchcock

If an algebraic equation with real coefficients happens to have a pair of conjugate *pure imaginary* roots we can divide through by a real factor $x^2 + q_1$ where $q_1$ is positive. If not, and if we call the real part of one pair of conjugate complex roots $r_1$, we can suppose that by Horner's method all the roots have their real parts reduced by the amount $r_1$. We shall need a test, however, to know when this has been accomplished.

Such a test is quite simple. Suppose there actually is a real factor of the form $x^2 + q$. Let $n = 2m$ and write the equation in the form

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0. \tag{1}$$

Divide through by $x^2 + q$. The remainder is

$$(a_1 - a_3 q + a_5 q^2 - \cdots \pm a_{n-1} q^{m-1})x + (a_0 - a_2 q + a_4 q^2 - \cdots \pm a_{n-2} q^{m-1} \mp q^m).$$

If this remainder is zero we have simultaneously

$$a_1 - a_3 q + a_5 q^2 - \cdots \pm a_{n-1} q^{m-1} = 0 \tag{2}$$

and

$$a_0 - a_2 q + a_4 q^2 - \cdots \pm a_{n-2} q^{m-1} \mp q^m = 0. \tag{3}$$

Hence the equations (2) and (3) must have a common solution $q = q_1$. This is the same as saying that the left members of these two equations have a common factor $q - q_1$, in which case the left side of (1) will have a factor $x^2 + q_1$. We can find the common factor $q - q_1$, if it exists, by the usual process of finding the greatest common divisor of two polynomials.

If, as of course will usually happen, there is no common factor the process of finding the greatest common divisor will yield a remainder which is a real number. This remainder will, in general, have opposite signs according to whether, by Horner's method, we have reduced the real parts of the roots not quite enough or a little too much.

As an example take the equation of the eighth degree

$$x^8 - 3.012\,x^7 + 3.225\,x^6 + 1.021\,x^5 + 6.986\,x^4 - 21.887\,x^3 + 8.110\,x^2 + 5.901\,x + 23.889 = 0 \tag{4}$$

The test equations (2) and (3) are then

$$3.012\,q^3 + 1.021\,q^2 + 21.887\,q + 5.901 = 0 \qquad (5)$$

$$q^4 - 3.225\,q^3 + 6.986\,q^2 - 8.110\,q + 23.889 = 0. \qquad (6)$$

We wish to find what is the sign of the remainder when we go through the process of finding the greatest common divisor of the left sides of these two equations. For convenience we begin by dividing (5) by $+3.012$. (Note, however, that we must never divide through by a *negative* number, since this might alter the sign of the remainder). We then go through the usual process of finding the greatest common divisor and the remainder comes out $+1.3$ approximately.

By inspection of the given equation (4) we can see that there must be at least one pair of complex roots whose real part is negative, for if not the signs would fall alternately plus and minus. So we try adding $+1$ to all the roots by Horner's method. This gives the new equation

$$x^8 - 11.012\,x^7 + 52.309\,x^6 - 137.581\,x^5 + 225.676\,x^4 - 265.541\,x^3 + 245.104\,x^2 - 147.253\,x + 61.187 = 0. \qquad (7)$$

The new test equations are

$$11.012\,q^3 - 137.581\,q^2 + 265.541\,q - 147.253 = 0 \qquad (8)$$

$$q^4 - 52.309\,q^3 + 225.676\,q^2 - 245.104\,q + 61.187 = 0 \qquad (9)$$

On finding the greatest common divisor of the left sides, the remainder comes out $-.08$ approximately. This proves that there is at least one pair of roots with real part between zero and $-1$, since the sign of the remainder has been changed.

Continuing in this way we might estimate the real parts of all the roots, locating them as nearly as we wish, much as, in the usual process of finding *real* roots by Horner's method we first locate a root approximately by means of trial values.

In the present case we note that the smallness of the remainder, namely $-.08$, in the last step described, seems to show that one pair of roots has real part not far from $-1$. In finding the greatest common divisor of the left members of (8) and (9) the last divisor was $-q + 1.198$ as the reader may verify. If the numerical remainder had been zero a pair of roots would have had real part exactly $-1$ and imaginary parts $\pm\sqrt{-1.198}$, that is $\pm\,1.095j$ where $j = \sqrt{-1}$. We conclude that a pair of roots are approximately $-1 \pm 1.095j$.

I shall take advantage of this circumstance to show how, when a pair

of roots is approximately known, more accurate values can be found with increasing rapidity the more accurate the approximation becomes.

Rule. Let an approximation to a pair of roots be $r \pm kj$, where $r$ and $k$ are real and $k$ is positive. Reduce the real parts of the roots of the given equation by the amount $r$ by Horner's method. Call the new equation with real parts thus reduced

$$x^n + A_{n-1}x^{n-1} + \cdots + A_1x + A_0 = 0. \tag{10}$$

Letting $k^2 = q$ calculate the following four quantities,

$$s = A_0 - A_2q + A_4q^2 - \cdots \pm A_{n-2}q^{m-1} \mp q^m, \tag{11}$$

$$t = k(A_1 - A_3q + A_5q^2 - \cdots \pm A_{n-1}q^{m-1}) \tag{12}$$

$$u = A_1 - 3A_3q + 5A_5q^2 - \cdots \pm (n-1)A_{n-1}q^{m-1}, \tag{13}$$

$$v = 2k(A_2 - 2A_4q + 3A_6q^2 - \cdots \pm mq^{m-1}). \tag{14}$$

Then

$$-\frac{su + tv + (tu - sv)j}{u^2 + v^2} \tag{15}$$

will in general be a very good approximation to the difference between the first approximation $r + kj$ and a true root.

I shall first apply this rule to the above example, and later explain its theory.

Taking $-1 \pm 1.095j$ as the first approximation, i.e. $r = -1$ and $k = 1.095$, with $q = 1.198$, the equation (10) becomes precisely the equation (7) already found. Calculating $s$, $t$, $u$, $v$ and then (15) we find

$$-.040 - .032j \tag{16}$$

for the value of (15). Adding this to the first approximation we have

$$-1.040 + 1.063j \tag{17}$$

as a considerably better approximation, i.e. a new and better value for $r + kj$.

To get greater accuracy we now go back to the original equation (4) and increase the real parts by $+1.04$, or, what is the same thing, increase the real parts of (7) by $+.04$. With the new coefficients thus obtained we find a new value for (15), which turns out to be

$$-.002869 + .007971j \tag{18}$$

and adding this value to (17) we have

$$-1.042869 + 1.070971j \tag{19}$$

as a root. This is probably correct, or nearly so, to four decimal places. In general it seems that the convergence is of much the same sort as in the usual Horner's method, for (15) will probably have nearly twice as many digits significant as the approximation $r + kj$ from which (15) was obtained, and the next place, while not usually correct, often has some significance.

In exceptional cases we may find that we have started to use the "Rule" too soon, and are not getting convergence. If so we must go back to the method of greatest common divisor and carry it further. This is likely to happen when there are two pairs of roots nearly equal in their real parts.

When we are getting convergence under the "rule" we need no longer test the remainder by the greatest common divisor method, but may do so as a check if we like. For instance, testing the remainder when $-1.043$ (taken from (19)) is used as approximate real part gives remainder less in numerical value than .001.

The explanation of the "rule" is as follows. If, starting with (10), we use Horner's method to reduce the *imaginary* part of the root $r + kj$ by the amount $kj$ we find the first remainder to be $s + tj$ and the second remainder to be $u + vj$. This is the same as saying that if the left member of (10) is called $f(x)$ then

$$f(x' + kj) = f(kj) + x'f'(kj) = s + tj + x'(u + vj) = 0$$

if we neglect the square of $x'$. Solving for $x'$ and rationalizing the denominator we obtain (15).

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# ÜBER EULER-LINIEN UNENDLICHER GRAPHEN

Von P. Erdös, T. Grünwald und E. Vázsonyi in Budapest

## Einleitung

1,1. Man versteht unter der Euler-Linie eines unendlichen Graphen eine Folge der Kanten, welche eine jede Kante des Graphen einmal und nur einmal enthält. Für endliche Graphen gilt der folgende Satz von Euler:[1] Ein aus endlich vielen Kanten bestehender Graph besitzt eine geschlossene Euler-Linie dann und nur dann, wenn

I. der Graph zusammenhängend,

II. geraden Grades ist.[2]

In seinem vorher zitierten Werke stellt D. König folgendes Problem: was ist die notwendige und hinreichende Bedingung dafür, dass ein unendlicher Graph eine Euler-Linie besitzt. Wir verstehen unter einer Euler-Linie eines unendlichen Graphen eine in beiden Richtungen unendliche Folge der Kanten, $\cdots, P_{-1}P_0, P_0P_1, \cdots$ in der eine jede Kante des Graphen genau einmal vorkommt.

Unsere Arbeit enthält die Lösung dieses Problems.

Es ist evident, dass ein unendlicher Graph nur dann eine Euler-Linie besitzen kann, wenn er

$T_1$. zusammenhängend ist,

$T_2$. die Anzahl der Kanten abzählbar ist,

$T_3$. enthält keinen einzigen Knotenpunkt ungeraden Grades.[3] Dies sind notwendige Bedingungen, die wir triviale Bedingungen nennen. In den folgenden Untersuchungen wird vorausgesetzt, dass der Graph diese Bedingungen erfüllt. Es ist leicht zu sehen, dass diese trivialen Bedingungen nicht hinreichend sind (Fig. 1.) stellt uns z.B. einen Graphen dar, welcher die trivialen Bedingungen erfüllt und doch keine Euler-Linie besitzt.

Wir nennen die zusammenhängenden Graphen, in welche der Graph

[1] S. z. B. D. König, Theorie der endlichen und unendlichen Graphen. Leipzig, 1936, p. 20.

[2] Ein Graph ist geraden Grades, wenn die Anzahl der sich in einem jeden Knotenpunkt treffenden Kanten gerade ist. Der Grad eines Knotenpunktes bedeutet die Anzahl der Kanten, die sich in diesem Punkte treffen.

[3] d. h. die Knotenpunkte der Graphen sind entweder geraden oder unendlichen Grades.

durch Weglassung eines endlichen Teilgraphen zerfällt, Komponente des Graphen und beweisen in §1, folgenden

*Satz*: Wenn ein unendlicher Graph $G$ eine Euler-Linie bezitzt, dann besteht $G - g$[4] für einen jeden endlichen $g \subset G$ aus höchstens 2 unendlichen Gliedern. In späteren bezeichnen wir diese Bedingung durch $E_1$.

Setzen wir nun voraus, dass die Notwendigkeit von $E_1$ schon bewiesen ist, und fragen ob es mit den trivialen Bedingungen zusammen hinreichend ist (Fig. 2) zeigt unmittelbar, dass die Antwort verneinend ist.

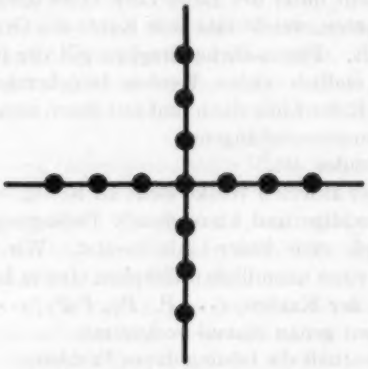


FIG. 1

FIG. 2

In §2 werden wir aber zeigen, dass wenn der unendliche Graph die trivialen Bedingungen erfüllt und für jedes endliche $g \subset G$, $G - g$ eine einzige unendliche Komponente bezitzt (Bedingung $E$) der Graph $G$ eine Euler-Linie hat.

Es ist aber aus (Fig. 3) klar, dass Bedingung $E$ nicht notwendig ist.

Bedingung $E$ ist daher hinreichend aber nicht notwendig, $E_1$ hingegen notwendig, aber nicht hinreichend: $E$ verlangt zu viel, $E_1$ zu wenig. Nun leiten wir aus $E$ eine notwendige Bedingun $ab$, indem wir nicht

[4] Der Graph $G - g$ bedeutet den Graph, welcher aus $G$ durch Weglassung von $g$ entsteht. Bemerkt sei dabei, dass wenn sämmtliche Kanten eines Knotenpunktes weggelassen werden, auch der betreffende Knotenpunkt wegzulassen ist. Die Zeichen $g \subset G$ und $P \subset G$ bedeuten, dass $g$ ein Teilgraph von $G$ beziehungsweise $P$ ein Knotenpunkt von $G$ ist.

verlangen dass im Falle eines jeden $g \subset G$ der graph $G - g$ eine einzige unendliche Komponente habe, sondern nur für graphen $g$ geraden grades.

Also beweisen wir -und das ist der eigentliche Gegenstand dieser Arbeit- folgendes.

*Satz*: Ein unendlicher Graph $G$ besitzt eine Euler-Linie dann und nur dann, wenn er die trivialen Bedingungen $T_1$, $T_2$, $T_3$, erfüllt und ausserdem.

$E_1$. im Falle eines beliebigen endlichen $g \subset G$ der Graph $G - g$ höchstens 2 unendliche Komponente hat.

$E_2$. wenn $g \subset G$ einen beliebigen, endlichen Graph geraden Grades bedeutet, der Graph $G - g$ eine einzige unendliche Komponente enthält.

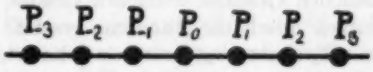



FIG. 3

1,2. Bevor wir aber unsern Satz beweisen, ersetzen wir die Bedingungen $E_1$ und $E_2$ durch andere, die etwas weniger fordern. Wir wollen nämlich voraussetzen, dass unser Satz schon bewiesen ist. Dann, da $G$ eine Euler-Linie besitzt, können wir zu einem jeden endlichen Graph $g \subset G$ eine Kantenzug[5] $z$ mit $g \subset z \subset G$ bestimmen. Wenn aber der Graph $G - z$ höchstens 2 unendliche Komponente besitzt, dann ist es mit Rücksicht auf $G - g \supset G - z$, unmittelbar klar, dass auch $G - g$ höchstens 2 unendliche Komponente hat. Es wäre daher denkbar, dass $E_1$ und $E_2$ eine hinreichende Bedingung darstellten, wenn man anstatt den Graphen $g$ Kantenzüge nimmt. Dies ist auch wirklich der Fall. Man kann sogar auch für die Kantenzüge etwas vorschreiben, Wir werden später sehen, dass sogar die folgenden Bedingungen hinreichend sind.

$E_1'$. Es existiert ein Knotenpunkt $Q \subset G$[6] so, dass für einen beliebigen

[5] Unter Kantenzug eines Graphen versteht man eine endliche Folge $P_1P_2$, $P_2P_3$, $\cdots$, $P_{n-1}P_n$ der Kanten, in der eine Kante höchstens einmal vorkommt. (Ein Kantenzug ist daher immer ein endlicher Graph.) Wenn $P_1 = P_n$ dann ist der Kantenzug geschlossen, er ist hingegen offen, wenn $P_1 \neq P_n$. (Die Euler-Linie eines endlichen Graphen ist daher ein geschlossener Kantenzug, der sämtliche Kanten des Graphen enthält.)

[6] Dieser Knotenpunkt des Graphen heisst ausgezeichneter Punkt. Diese Auszeichnung ist aber nicht im Graph selbst, sondern nur in der Bedingung enthalten, denn wenn der Graph einen ausgezeichneten Punkt besitzt, dann kann man alle seine Knotenpunkte für ausgezeichnet betrachten. Denn wenn

Kantenzug $z$ mit $Q \subset z \subset G$, der Graph $G - z$ höchstens zwei unendliche Komponenten besitzt.

$E_2'$. für einen beliebigen geschlossenen Kantenzug $z$ mit $Q \subset z \subset G$, besitzt der Graph $G - z$ genau eine unendliche Komponente.

Nun ist der Gedankengang unseres Beweises der Folgende. Wir beweisen, dass $E_1$. und $E_2$. notwendige Bedingungen sind, die Bedingungen $E_1'$ und $E_2'$ sind daher auch notwendig, da sie weniger verlangen. Dann beweisen wir, dass $E_1'$. und $E_2'$. hinreichend sind, woraus unmittelbar folgt, dass auch $E_1$. und $E_2$ hinreichend sind, da sie mehr verlangen.

1,3. Wir wollen endlich hervorheben, was die trivialen Bedingungen $T_1$, $T_2$ und $T_3$ für eine wichtige Klasse der unendlichen Graphen, für die sogenannten unendlichen Graphen endlichen Grades,[7] bedeuten. Bei diesen wird $T_3$ einfach durch die Annahme ersetzt, dass der Graph geraden Grades ist. $T_2$ bleibt weg, denn wie bekannt, sind die Kanten eines zusammenhängenden Graphen endlichen Grades abzählbar.[8] Die trivialen Bedingungen verlangen daher nur, dass der Graph *zusammenhängend* und *geraden* Grades sei.

## §1. Beweis der Notwendigkeit der Bedingungen $E_1$ und $E_2$

Wir setzen voraus, das der Graph $G$ eine Euler-Linie besizt d.h. es existiert eine Kantenfolge

$$\cdots, P_{-2}P_{-1}, P_{-1}P_0, P_0P_1, P_1P_2, \cdots$$

welche jede Kante einmal und nur einmal enthält. Es sei $g \subset G$ ein beliebiger endlicher Graph. Auf alle Fälle gibt es zwei ganze Zahlen $\mu$ und $\nu$, so dass für den Kantenzug

$$Z = (P_\mu P_{\mu+1}, P_{\mu+1}P_{\mu+2}, \cdots, P_{\nu-1}P_\nu) \text{ mit } P_\mu \neq P_\nu$$

$g \subset Z$ gilt. Es sei ferner

$$E_{-1} = (\cdots, P_{\mu-2}P_{\mu-1}, P_{\mu-1}P_\mu),$$
$$E_{+1}' = (P_\nu P_{\nu+1}, P_{\nu+1}P_{\nu+2}, \cdots),^9$$

---

der Graph $E_1'$, und $E_2'$ erfüllt, so besitzt er eine Euler-Linie, erfüllt daher auch die notwendigen Bedingungen $E_1$ und $E_2$. Also erfüllt $G - g$ die gewünschten Bedingungen für einen beliebigen endlichen Graph $g$.

[7] Ein Graph ist von endlichem Grade, wenn sich in einem jeden seiner Knotenpunkte endlich viele Kanten treffen.

[8] D. König, l.c. p. 79.

[9] Die Bezeichnung bedeutet, dass $E_{-1}$ und $E_{+1}$ einseitige unendliche Euler-Linien sind.

daher

$$G = E_{-1} + Z + E_{+1}.^{10}$$

Da $G - g \supset G - Z = E_{-1} + E_{+1}$, kann eine Komponente des Graphen $G - g$ nur dann unendlich sein, wenn sie $E_{-1}$ oder $E_{+1}$ enthält. Dies bedeutet aber, dass dieser Graph höchstens 2 unendliche Komponenten besitzt, also erfüllt $G$ die Bedingung $E_1$.

Jetzt betrachten wir die Bedingung $E_2$. Wir beweisen, dass wenn $g \subset G$ geraden Grades ist, so hat der Graph $G - g$ ein einziges unendliches Glied. Wir nehmen $Z$ wie oben und konstruieren einen Kantenzug $z$ mit

$$P_\nu, P_\mu \subset z \subset G - g, Z$$

(Wo $Z = (P_\mu P_{\mu+1}, \cdots, P_{\nu-1}P_\nu)$, $P_\mu \neq P_\nu$.) Die Konstruktion geschieht folgendermassen: Der Knotenpunkt $P_\mu$ ist in $Z$ ungeraden Grades, während er im Falle $P_\mu \subset g$ in $g$ vom geraden Grade ist. Also ist der Knotenpunkt $P_\mu$ im Graph $Z - g$ ungeraden Grades, woraus die Existenz einer Kante $P_\mu P_{i_1} \subset Z - g$ folgt. Für $P_{i_1} = P_\nu$, setzen wir $z = P_\mu P_{i_1}$. Wenn aber $P_{i_1} \neq P_\nu$, dann ist es mit Rücksicht darauf, dass $P_{i_1}$ in dem Kantenzug $Z$ geraden Grades ist und für $P_{i_1} \subset g$ auch in $g$ geraden Grades ist, auch in $Z - g$ geraden Grades. Also ist $P_{i_1}$ im Graph $Z - g - P_\mu P_{i_1}$ ungeraden Grades, woraus folgt, dass es eine Kante $P_{i_1}P_{i_2} \subset Z - g - P_\mu P_{i_1}$ existiert. Wenn $P_{i_2} = P_\nu$, so ist

$$z = (P_\mu P_{i_1}, P_{i_1}P_{i_2});$$

wenn hingegen $P_{i_2} \neq P_\nu$ und $P_{i_2} \neq P_\mu$ so existiert -ebenso wie früher- eine Kante

$$P_{i_2}P_{i_3} \subset Z - g - P_\mu P_{i_1} - P_{i_1}P_{i_2}$$

Wir können eine solche Kante auch dann finden, wenn $P_{i_2} = P_\mu$. (Denn der Knotenpunkt $P_{i_2} = P_\mu$ ist in $Z$ ungeraden Grades, im Graph $(P_\mu P_{i_1}, P_{i_1}P_{i_2})$ hingegen geraden Grades, ferner für $P_\mu \subset g$ auch in $g$ geraden Grades. Der Knotenpunkt $P_{i_2} = P_\mu$ ist daher im Graph

$$Z - g - P_\mu P_{i_1} - P_{i_1}P_{i_2}$$

ungeraden Grades; woraus folgt, dass es eine Kante

$$P_{i_2}P_{i_3} \subset Z - g - P_\mu P_{i_1} - P_{i_1}P_{i_2}$$

[10] Die Summe zweier keine gemeinsame Kante enthaltenden Graphen bedeutet den Graph, der von den Kanten beider Graphen gebildet wird. $z \subset Z, Z'$ bedeutet (unten) $z \subset Z$ und $z \subset Z'$.

gibt.) Wir können dieses Verfahren solange fortsetzen, bis wir den Knotenpunkt $P_r$ erreicht haben. Wir erhalten so einen Kantenzug, welche unseren Erforderungen entspricht, da sie $P_s$ mit $P_r$ verbindet und da aus $z \subset Z - g$ und $Z \subset G$ folgt, dass $z \subset G - g, Z$.

$(G - Z)^{11} + z = E_{-1} + z + E_{+1}$ ist aber ein zusammenhängender Graph mit

$$(G - Z) + z \subset G - g,$$

woraus tatsächlich folgt, dass der Graph $G - g$ nur eine einzige unendliche Komponente besitzt.

Also haben wir die Notwendigkeit der Bedingungen $E_1$ und $E_2$ bewiesen, woraus natürlich auch die Notwendigkeit der weniger verlangenden Bedingungen $E_1'$ und $E_2'$ folgt.

## §2. Beweis, dass Bedingung $E$ hinreichend ist

2,1. Wir verfolgen hier, den in der Einleitung angegebenen Weg: wir beweisen nämlich, dass auch schon die weniger verlangenden Bedingungen $E_1'$. und $E_2'$. hinreichend sind. Entsprechend beweisen wir nicht $E$ sondern zeigen, dass der Graph $G$ eine Euler-Linie besitzt, wenn er ausser den trivialen Bedingungen noch folgende Bedingung erfüllt:

$E'$. Es existiert ein Knotenpunkt[12] $Q \subset G$ derart, dass für einen jeden Kantenzug $Q \subset z \subset G$ der Graph $G - z$ eine einzige unendliche Komponente hat.

2,2. Der Beweis wird in 2 Schritten geführt:

Wir numerieren die Kanten des Graphen $G$ beliebig und konstruieren einen geschlossenen Kantenzug $Z_0$ so, dass

1. $Q \subset Z_0$,

2. der Kantenzug enthält die Nummer 1 führende Kante[13] des Graphen $G$,

3. Der Graph $G - Z_0$ erfüllt sowohl die trivialen Bedingungen wie auch $E'$.[14] und zwar so dass der ausgezeichnete Knotenpunkt $Q_1 \subset Z_0$ ist.

Wir nehmen einen Kantenzug $z_1 \subset G$ welcher aus dem ausgezeichneten Knotenpunkt $Q = Q_0$ ausgeht und die Kante 1 enthält. Nach $E'$

[11] Der Graph $G - Z$ und die Kantenlinie $z$ haben, wegen $z \subset Z$ keine gemeinsame Kante.

[12] Den Knotenpunkt $Q$ nennt man "ausgezeichneten Knotenpunkt" Vgl. Fussnote 6.

[13] Der Kürze halber nennen wir diese Kante 1.

[14] Dies ist so zu verstehen, dass man in $E'$. $G$ durch $G - Z_0$ ersetzt.

enthält der Graph $G - z_1$ nur eine einzige unendliche Komponente. Wir fügen die übrigen (endlichen) Komponenten zu $z_1$ und erhalten einen endlichen Graph $g_1$.[15]

Es ist unschwer zu sehen, dass die Anzahl der Knotenpunkte ungeraden Grades in $g_1$ entweder 0 oder 2 ist. Betrachten wir nämlich zuerst den Knotenpunkt $P \subset g_1$, $P \not\subset G - g_1$. Dann ist $P$ in $g_1$ vom selben Grade als in $G$, d.h. er ist mit Rücksicht auf die Endlichkeit von $g_1$, geraden Grades. Wenn aber $P \subset g_1$, $G - g_1$, dann ist $P \subset z_1$. Denn wenn $P \not\subset z_1$ wäre, dann wäre $P$, mit Rücksicht darauf, dass $g_1$ aus $z_1$ durch Zufügung der endlichen Glieder des Graphen $G - z_1$ entstanden ist, in $g_1$ vom selben Grade wie in $G$; und so würde entgegen den Voraussetzungen $P \not\subset G - g_1$ gelten. Wir behaupten ferner, dass in diesem Falle auch $P_1 \not\subset g_1 - z_1$ gilt. Denn $P \subset g_1$, $g_1 - z_1$ würde bedeuten, dass bei der Zufügung der Glieder von $G - z_1$ an $z_1$ auch Kanten, die sich in $P$ treffen hinzugefügt worden sind. Dann hätten wir aber sämtliche sich in $P$ treffende Kanten von $G - z_1$ an $z_1$ hinzugefügt, d.h. der Grad von $P$ würde in $g_1$ und $G$ derselbe sein. Es wäre daher $P \not\subset G - g_1$ ein offenbarer Widerspruch. Aus $P \subset z_1$ und $P \not\subset g_1 - z_1$ folgt aber, dass sich in $P$ nur Kanten des $z_1$ treffen, d.h. $P$ ist in $g_1$ vom selben Grade als in $z_1$. Wenn daher $z_1$ einen geschlossenen Kantenzug darstellt, dann hat $g_1$, überhaupt keinen Knotenpunkt vom ungeraden Grade. Dasselbe gilt, wenn die endpunkte des offenen Kantenzuges $z$ die Punkte $P_1$, $P_2 \not\subset G - g_1$ sind. Wenn aber die Endpunkte des offenen Kantenzuges $z_1$ die Punkte $P_1$, $P_2 \subset G - g_1$ sind, dann besitzt $g_1$ in diesen beiden Punkten einen Knotenpunkt ungeraden Grades. Damit haben wir unsere Behauptung für $g_1$ bewiesen.

Wenn $g_1$ geraden Grades ist, dann bezeichnen wir ihn mit $g$. Wenn er hingegen zwei Knotenpunkte ungeraden Grades $P_1$, $P_2$ besitzt, dann gilt -nach dem Vorhergesagten- $P_1$, $P_2 \subset G - g_1$ Da $G - g_1$ zusammenhängend ist, so muss er einen die Knotenpunkte $P_1$ und $P_2$ verbindenden Kantenzug besitzten. Wir fügen nun diesen Kantenzug zu $g_1$ und erhalten eine endlichen Graph $g_2$. Da $g_2$ geraden Grades ist, können wir ihn -nach dem in der Einleitung angeführten Eulerschen Satze- für einen geschlossenen Kantenzug $z_2$ betrachten. Infolge von $Q \subset z_2$ besitzt der Graph $G - z_2$—nach $E'$—nur eine einzige unendliche Komponente. Wir fügen die endlichen Glieder des Graphen $G - z_2$ zu $z_2$ und erhalten einen endlichen Graph geraden Grades $g$. Dieser

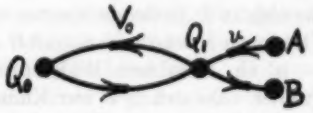
[15] $g_1$ ist endlich, da die Anzahl der endlichen Komponenten des Graphen $G - z_1$ notwendigerweise endlich ist.

Graph ist geraden Grades, da er nur dort einen Knotenpunkt ungeraden Grades haben könnte, wo auch $z_2$ einen solchen Knotenpunkt hat (dies ist ebenso zu beweisen, wie wir es früher bewiesen haben, dass der Graph $g_1$ nur dort einen Knotenpunkt ungeraden Grades haben kann, wo auch $z_1$ einen solchen besitzt). Da aber $z_2$ ein geschlossener Kantenzug ist, hat er keinen einzigen Knotenpunkt ungeraden Grades. Auf alle Fälle haben wir auf diese Weise einen endlichen Graph geraden Grades $g$, für welchen der Graph $G - g$ zusammenhängend ist.

Nach dem Eulerschen Satz können wir den Graph $g$ als einen geschlossenen Kantenzug $Z_0$ auffassen. Wir behaupten, dass $Z_0$ die gewünschten Bedingungen erfüllt. Es ist nämlich $Q_0 \subset Z_0$ und die Kante 1 ist in $Z_0$ enthalten, also erfüllt $Z_0$ Bedingungen 1. und 2.

Das auch 3. erfüllt wird, sehen wir folgendermassen ein:

$T_1$. ist erfüllt, da $G - Z_0 = G - g$




$$V_0 + v = AQ + Q_1Q_0 + Q_0Q_1 + Q_1B$$

FIG. 4

$T_2$. , $T_3$. sind erfüllt, da der Graph $G - Z_0$ aus $G$ durch Weglassung eines endlichen Graphen geraden Grades enstanden ist.

$E'$. wird dadurch erfüllt, dass man ein $Q_1 \subset G - Z_0$, $Z_0$ zum ausgezeichneten Knotenpunkt wählt. Denn für einen solchen haben wir nur zu beweisen, dass der Graph $G - Z_0 - z$ für einen beliebigen $Q_1 \subset z \subset G - Z_0$ Kantenzug nur eine einzige unendliche Komponente besitzt. Wie aber aus (Fig. 4) zu ersehen ist bildet der Kantenzug $z$ mit dem geschlossenen Kantenzug $Z_0$ zusammen einen Graph, der mit Rücksicht auf $Q_1 \subset z$, $Z_0$, für einen Kantenzug $z'$ zu betrachten ist. Da $Q_0 \subset z'$ folgt aus $E'$, dass der Graph $G - z'$ eine einzige unendliche Komponente hat. Es ist aber

$$G - z' = G - Z_0 - z;$$

wir haben daher bewiesen, dass $Z_0$ das Erwünschte leistet.

2,3. Der Graph $G - Z_0$ erfüllt die trivialen Bedingungen $T_1$. , $T_2$. , $T_3$ ferner auch $E'$. und zwar so, dass man einen Knotenpunkt $Q_1 \subset Z_0$,

$G - Z_0$ zum ausgezeichneten Knotenpunkt wählen kann. Wir können daher die frühere Konstruktion für den Graph $G - Z_0$ wiederholen. Durch die Wiederholung erhalten wir einen geschlossenen Kantenzug $Q_1 \subset Z_1 \subset G - Z_0$, welcher die Kante niedrigster Nummer des Graphen $G - Z_0$ enthält. Der Graph $G - Z_0 - Z_1$ erfüllt wieder $T_1$, $T_2$, $T_3$ und auch $E'$ und zwar so, dass man einen Knotenpunkt $Q_2 \subset G - Z_0 - Z_1$, $Z_1$ zum ausgezeichneten Punkte wählt. Wir setzen dieses Verfahren fort und erhalten die unendliche Folge $Z_0$, $Z_1$, $Z_2$, $\cdots$ geschlossener Kantenzüge ohne gemeinsame Kante. Dieser unendliche Kantenzug enthält alle Kanten des Graphen $G$ und $Q_i$, $Q_{i+1} \subset Z_i$. Der geschlossene Kantenzug $Z_0$ ist aber auch für einen aus $Q_1$ ausgehenden und ebendorthin zurückkehrenden Kantenzug $Z_0'$, während der geschlossene Kantenzug $Z_i$ in zwei die Punkte $Q_i$ und $Q_{i+1}$ verbindende offene

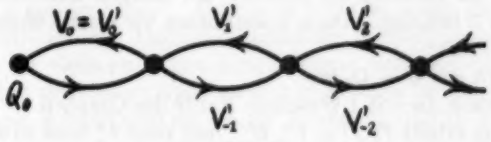



FIG. 5

Kantenzüge $Z'_{-i}$ und $Z'_{+i}$ ($i = 1, 2, 3, \cdots$) zu zerlegen ist. Die Folge der zusammenschliessenden Kantenzüge

$$\cdots, Z'_{-2}, Z'_{-1}, Z'_0, Z'_1, Z'_2, \cdots$$

gibt die gesuchte Euler-Linie des Graphen (Fig. 5).

## §3. Über einseitig unendliche Euler-Linien

3,1. Wir definieren als einseitig unendliche Euler-Linie eines unendlichen Graphen eine einseitig unendliche Folge seiner Kanten

$$P_0P_1, P_1P_2, P_2P_3, \cdots$$

welche eine jede Kante des Graphen einmal und nur einmal enthält. Bei dem Beweis, dass die Bedingungen $E'_1$. und $E'_2$ hinreichend sind, brauchen wir ein Kriterium, durch welches wir entscheiden können, ob ein unendlicher Graph eine einseitig unendliche Euler-Linie besitzt.

In diesem § werden wir zeigen, dass der unendliche Graph $G$ eine einseitig unendliche Euler-Linie besitzt, wenn er folgende Bedingungen erfüllt

I. $T_1$.,
II. $T_2$.,
III. $E$.,
IV. und noch folgende Bedingung

$T^*$, welche besagt: wenn $G$ einen Knotenpunkt $Q$[16] ungeraden Grades hat, so ist dieser der einzige Knotenpunkt ungeraden Grades; wenn $G$ keinen Knotenpunkt ungeraden Grades hat, so gibt es in ihm wenigstens einen Knotenpunkt $Q$ unendlichen Grades.[17]

Wie in §2 ersetzen wir $E$. durch eine weniger verlangende Bedingung. $E''$. Im Falle eines aus, durch $T^*$ ausgezeichneten Knotenpunkt $Q$ ausgehenden Kantenzuges $z \subset G$ hat der Graph $G - z$ eine einzige unendliche Komponente.[18]

3,2. Die Überlegungen, durch welche wir unser Kriterium beweisen, sind den in §2. verwendeten ähnlich. Zu allererst numerieren wir die Kanten von $G$ beliebig. Dann konstruieren wir einen Offenen Kantenzug $Z_0 \subset G$ mit:

(1) er geht aus $Q = Q_0$ aus

(2) er enthält die mit 1 versehene Kante des Graphen

(3) $G - Z_0$ erfüllt $T_1$, $T_2$, $T^*$, $E''$[19] und zwar $T^*$ wird so erfüllt, dass der ausgezeichnete Punkt der von $Q_0$ verschiedene Endpunkt des offenen Kantenzuges $Z_0$ ist.

$Z_0$ wird folgendermassen konstruiert:

Wir betrachten einen aus dem ausgezeichneten Knotenpunkt ausgehenden offenen Kantenzug $z_1$ des $G$ welcher Kante 1 enthält.[20] Wir

[16] Im Folgenden wollen wir diesen Knotenpunkt ausgezeichneten Punkt des Graphen nennen.

[17] Es wäre unschwer zu beweisen, dass die Bedingungen auch notwendig sind, Da wir aber diesen Satz nicht brauchen, werden wir ihn auch nicht beweisen.

[18] $E''$ unterscheidet sich von $E'$ in zwei Punkten. Erstens wird der "ausgezeichnete" Punkt durch $T^*$ tatsächlich ausgezeichnet (Vergleiche Fussnote 6.), zweitens ist der Kantenzug $z$ nicht ein beliebiger durch $Q$ gehender, sondern aus $Q$ ausgehender Kantenzug.

[19] Dies ist wieder so zu verstehen, dass in den Kriterien $G$ durch $G - Z_0$ zu ersetzen ist.

[20] Ein solcher Kantenzug existiert. Denn, da $G$ zusammenhängend ist, ist in ihm notwendigerweise ein aus $Q_0$ ausgehender Kantenzug zu finden, welcher die Kante 1 enthält. Wenn dieser Kantenzug geschlossen ist, kann er als Graph geraden Grades sämtliche sich in $Q_0$ treffende Kanten des $G$ nicht erschöpfen (die Anzahl dieser Kanten ist nicht gerade). Man kann daher zu diesem geschlossenen Kantenzug eine aus $Q_0$ entspringende Kante des $G$ hinzufügen und dadurch erhält man schon aus dem geschlossenen Kantenzug den gewünschten offenen Kantenzug.

bezeichnen den von $Q_0$ verschieden Endpunkt dieses Kantenzuges durch $Q_1$. Da $z_1$ aus $Q_0$ entspringt hat der Graph $G - z_1$ nach $E''$ nur eine einzige unendliche Komponente. Wir fügen seine endlichen Komponenten zu $z_1$ und erhalten dadurch den endlichen Graph $g_1$. Wir behaupten, dass $Q_0$ und $Q_1$ ungeraden Grades, alle übrigen Knotenpunkte des Graphen $g_1$ hingegen geraden Grades sind. Wir argumentieren hier genau wie in 2,2. Es sei zuerst $P \subset g_1$, aber $P \not\subset G - g_1$. Dann ist $P$ in $g_1$ desselben Grades, wie in $G$, also ungeraden Grades in $g_1$ wenn $P = Q_0$ und geraden Grades, wenn $P \neq Q_0$ (der Grad kann nicht unendlich werden, da $g_1$, selbst endlich ist). Wenn hingegen $P \subset g_1$, $G - g_1$, dann ist $P \subset z_1$ und $P \not\subset g_1 - z_1$,[21] also ist der Grad von $P$ in $g_1$ und $z_1$ derselbe; also geraden Grades, wenn $P \neq Q_0$, $Q_1$ —, ungeraden Grades, wenn $P = Q_0$, $Q_1$. Also haben wir bewiesen, dass $Q_0$ auf alle Fälle ungeraden Grades ist, während wir dies für $Q_1$ nur im Falle $Q_1 \subset G - g_1$ gezeigt haben. Aber es ist leicht zu sehen, dass $Q_1$ immer ungeraden Grades ist. Denn die Anzahl der Knotenpunkte ungeraden Grades ist für einen endlichen Graph immer eine Gerade Zahl[22] also kann $Q_0$ nicht der einzige Knotenpunkt ungeraden Grades in $g_1$ sein.

Der Graph $g_1$ kann also als ein die Punkte $Q_0$ und $Q_1$ verbindender offener Kantenzug $Z_0$ betrachtet werden.[23] Wir behaupten, dass der offene Kantenzug $Z_0$ allen gewünschten Erforderungen entspricht. (1) und (2) erfüllt er offenbar, also müssen wir uns ausführlicher nur mit (3) beschäftigen.

Dass $G - Z_0$ die gewünschten Bedingungen erfüllt, sieht man folgendermassen:

$T_1$. ist erfüllt, da $G - Z_0 = G - g_1$

ebenso $T_2$., da die Kanten des $G$ abzählbar sind.

$T^*$. ist auf die gewünschte Weise erfüllt, denn erstens sind die Knotenpunkte $P \subset G - Z_0$, $P \not\subset Q_0$, $Q_1$, in $Z_0$ geraden Grades, in $G$ hingegen entweder geraden oder aber unendlichen Grades; also ist $P$ auch in $G - Z_0$ entweder geraden oder unendlichen Grades. Der Knotenpunkt $P = Q_0 \subset G - Z_0$ ist in $Z_0$ ungeraden Grades, in $G$ hingegen ungeraden oder unendlichen Grades. Wenn also der Graph $G - Z_0$ überhaupt einen Knotenpunkt ungeraden Grades hat, so kann dies nur der Knotenpunkt $Q_1$ sein. Tatsächlich, wenn $Q_1$ in $G$ endlichen d.h. geraden Grades ist, so ist er in $G - Z_0$ ungeraden Grades, da er in $Z_0$ ungeraden Grades ist. Wenn aber $Q_1$ in $G$ unendlichen Grades ist, so hat der

[21] Siehe ausführlicher in 2,2,.

[22] S. z. B. D. König, op. cit., h. 21.1.

[23] Siehe in 5.1.

Graph $G - Z_0$ überhaupt keinen Knotenpunkt ungeraden Grades; er hat aber in diesem Falle wenigstens einen Knotenpunkt unendlichen Grades und der Knotenpunkt $Q_1$ ist eben ein solcher Punkt.

$E''$ wird ebenfalls erfüllt. Wir haben nur zu zeigen, dass der Graph $G - Z_0 - z$ für einen jeden aus $Q_1$ ausgehenden Kantenzug $z \subset G - Z_0$ nur eine einzige unendliche Komponente hat. Die aus $Q_1$ ausgehenden keine gemeinsamen Kanten besitzenden Kantenzüge $Z_0$ und $z$ kann man aber als einen einzigen aus $Q_0$ ausgehenden Kantenzug $z'$ betrachten ($Z_0$ entspringt nämlich im Punkte $Q_0$), also, mit Rücksicht darauf, dass $G$ die Bedingung $E''$ erfüllt, hat der Graph $G - z' = G - Z_0 - z$ nur eine einzige unendliche Komponente.

3,3. Unsere Konstruktion ist für den Graph $G - Z_0$ zu wiederholen. Auf diese Weise gewinnen wir einen offenen Kantenzug $Z_1 \subset G - Z_0$,

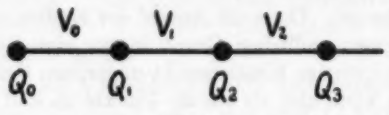



Fig. 6

dessen Endpunkte $Q_1$ und $Q_2$ sind, der die mit der kleinsten Nummer versehene Kante von $G - Z_0$ enthält und für den der Graph $G - Z_0 - Z_1$ die Bedingungen $T_1$, $T_2$, $T^*$ und $E''$ mit $Q_2$ als ausgezeichneten Punkt erfüllt. Durch widerholte Verwendung des Verfahrens erhalten wir eine unendliche Folge offener Kantenzüge

$$Z_0, Z_1, Z_2, \cdots,$$

die keine gemeinsamene Kanten haben und sich in den Knotenpunkten $Q_1, Q_2, \cdots$ aneinander anschliessen und die Kanten des Graphen $G$ erschöpfen. Die Folge dieser Kantenzug gibt die gesuchte einseitig unendliche Euler-Linie des Graphen (Fig. 6).

## §4. Beweis des Hauptsatzes

In diesem Paragraph werden wir beweisen, dass die Bedingungen $E_1'$ und $E_2'$ hinreichend sind.

In §2 haben wir gezeigt, dass wenn der Graph neben diesen Bedingungen auch $E'$ erfüllt, dann besitzt er eine Euler-Linie. Wir setzen jetzt voraus, dass dies nicht besteht, es existiert also ein Kantenzug $Q \subset z_1 \subset G$ so, dass der Graph $G - z_1$ aus zwei unendlichen Gliedern

besteht. Nach $E_2'$ ist $z_1$ sicherlich ein offener Kantenzug. Wir fügen jetzt die endlichen Glieder des Graphen $G - z_1$ zu $z_1$ und erhalten dadurch einen endlichen Graph $g_1$. Wir behaupten, dass $g_1$ genau zwei Knotenpunkte ungeraden Grades besitzt. [Genau wie in 2,2 können wir auch hier beweisen, dass $g_1$ entweder 2 Knotenpunkte ungeraden Grades hat oder selbst vom geraden Grade ist. Es ist aber unschwer zu sehen, dass $g_1$ nicht geraden Grades sein kann, denn wenn $g_1$ geraden Grades wäre, so könnten wir ihn für einen geschlossenen Kantenzug $z_2$ betrachten. Anderseits aus $z_2 = g_1 \supset z_1$ folgt, dass $G - z_2 \subset G - z_1$. Also wenn $G - z_1$ zwei unendliche Komponenten hätte, hätte auch $G - z_2$ zwei unendliche Komponenten (mehr nicht, denn $G$ erfüllt $E_1'$). Dies ist aber unmöglich, denn $G$ erfüllt $E_2'$ d. h. wenn $Q \subset z_2$ ein geschlossener Kantenzug ist, hat $G - z_2$ eine einzige unendliche Komponente. Der Graph $g_1$ kann daher für einen offenen Kantenzug $Z$ mit den Endpunkten $Q_{-1}$ und $Q_{+1}$ betrachtet werden. Nun besteht der Graph $G - Z$ aus zwei unendlichen Komponenten $G_{-1}$ und $G_{+1}$. Wir haben daher

$$G = G_{-1} + Z + G_{+1}$$

Wir behaupten, dass die Graphen $G_{-1}$ und $G_{+1}$ beide einseitig unendliche Euler-Linien besitzen, die aus den Endpunkten $Q_{-1}$ beziehungsweise $Q_{+1}$ ausgehen. Wir wenden nämlich die Kriterien des §3 auf $G_{-1}$ an. $G_{-1}$ erfüllt die Bedingungen $T_1$ und $T_2$. $T^*$ wird auch erfüllt und zwar mit dem ausgezeichneten Knotenpunkt $Q = Q_{-1}$. [Der Knotenpunkt $P \subset G_{-1}$, $P \not\subset G - G_{-1}$ hat nämlich in $G_{-1}$ denselben Grad wie in $G$ er kann daher nicht ungeraden Grades sein. Anderseits haben wir für den Knotenpunkt $P \subset G_{-1}$, $G - G_{-1}$ (mit Rücksicht darauf, dass $G_{-1}$ und $G_{+1}$ keinen gemeinsamen Knotenpunkt haben), $P \not\subset G_{+1}$ ferner, mit Rücksicht darauf, dass

$$G - G_{-1} = Z + G_{+1}$$

$P \subset Z$. Daraus folgt aber, dass der Grad von $P$ in $G_{-1}$ gleich der Differenz der Grade des Knotenpunktes $P$ in $G$ und $Z$ ist. Da $P$ in $G$ entweder geraden oder unendlichen Grades ist, während $P \neq Q_{-1}$ in $Z$ geraden Grades ist, kann der Knotenpunkt $P \neq Q_{-1}$ in $G_{-1}$ nicht ungeraden Grades sein. Anderseits ist $Q_{-1}$ in $Z$ ungeraden Grades, also in $G_{-1}$ entweder ungeraden oder unendlichen Grades, je nachdem $Q_{-1}$ in $G$ endlichen (d. h. geraden) oder unendlichen Grades ist.]

Endlich erfüllt $G_{-1}$ auch $E''$. Wir werden nämlich zeigen, dass der Graph $G_{-1} - z$ für einen aus $Q_{-1}$ entspringenden Kantenzug $z \subset G_{-1}$ eine einzige unendliche Komponente besitzt. Die beiden, aus $Q_{-1}$

entspringenden keine gemeinsamen Kanten besitzenden Kantenzüge $z$ und $Q \subset Z$ können als ein Kantenzug $z' \supset Q$ aufgefasst werden. Da $G$ $E_1'$ erfüllt, hat der Graph $G - z'$ höchstens 2, und wegen $G - z' \subset G_{-1} + G_{+1}$ wenigstens 2, also genau 2 unendliche Komponenten. Die eine unendliche Komponente ist $G_{+1}$, der Graph $G - z' - G_{+1}$ hat daher genau ein einziges unendliches Glied. Es gilt aber

$$G - z' - G_{+1} = (G_{-1} + Z + G_{+1}) - (z + Z) - G_{+1} = G_{-1} - z.$$

Also haben wir bewiesen, dass $G_{-1}$ als eine, aus $Q_{-1}$ entspringende einseitig unendliche Euler-Linien $E_{-1}$ aufzufassen ist. Auf ähnliche Weise folgt, dass $G_{+1}$ als eine, aus $Q_{+1}$ entspringende einseitig unendliche Euler-Linien aufzufassen ist. Mit Rücksicht darauf, dass

$$G = G_{-1} + Z + G_{+1} = E_{-1} + Z + E_{+1}$$

stellen die zusammenschliessenden Linien $E_{+1}$, $Z$ und $E_{-1}$ die gesuchte Euler-Linie des Graphen $G$ (Fig. 7).

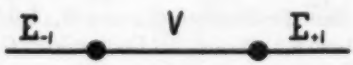



FIG. 7

## §5. Verwandte Probleme

In diesem § beschäftigen wir uns mit 3 von D. König gestellten Problemen.

5,1. Für endliche Graphen gilt der folgende Satz von Listing.[24]

Die Anzahl der Knotenpunkte ungeraden Grades eines endlichen Graphen sei $2n$.[25] Es existiert ein System der Kantenzüge $Z_1, Z_2, \cdots, Z_n$, welche eine jede Kante des gegebenen Graphen einmal und nur einmal enthält.

Durch die in dieser Arbeit verwendete Methode kann man diesen Satz auch auf unendliche Graphen übertragen. Es gilt nämlich folgender Satz: (dessen Beweis wir in dieser Arbeit nicht geben.)

*Die notwendigen und hinreichenden Bedingungen dafür dass zum unendlichen Graphen G ein endliches System* $Z_1, Z_2, \cdots Z_n$ *der endlichen,*

[24] D. König, op. cit., p. 22.

[25] Wie erwähnt ist die Anzahl der Knotenpunkte ungeraden Grades eines endlichen Graphen immer eine gerade Zahl.

*einseitig unendlichen, oder von beiden Seiten unendlichen Kantenzüge,*[36] *welche die Kanten der Graphen genau einmal enthalten existieren, sind die folgenden;*

1. $G$ erfüllt $T_2$,
2. $G$ enthält höchstens endlich viele Knotenpunkte ungeraden Grades.
3. es existiert eine Zahl $C$ so, dass die Anzahl der unendlichen Teile des Graphen $G - g$, für ein beliebiges endliches $g \subset G$ kleiner als $C$ ist.[37]

5,2. Für endliche Graphen gilt folgender Satz von Euler:

Die Kanten eines jeden endlichen zusammenhängenden Graphen sind durch eine Kantenfolge

$$P_0P_1, P_1P_2, \cdots, P_{n-1}, P_n, P_nP_0$$

zu erschöpfen, in welcher eine jede Kante des Graphen genau zweimal vorkommt.

Die Übertragung dieses Satzes auf unendliche Graphen lautet folgendermassen:

*Die notwendige und hinreichende Bedingung dafür, dass die Kanten des unendlichen Graphen G durch eine von beiden Seiten unendliche Kantenfolge*

$$\cdots, P_{-2}P_{-1}, P_{-1}P_0, P_0P_1, \cdots$$

*zu erschöpfen seien sind und zwar so, dass die Folge eine jede Kante des Graphen G genau zweimal enthält, dass G folgende Bedingungen erfüllt*

1. $T_1$
2. $T_2$
3. $E$[38]

Betrachten wir nämlich den Graph $G^*$, welcher aus Gentsteht, wenn man zu einer jeden Kante $K \subset G$ eine andere Kante $K'$ hinzufügt, welche dieselben 2 Punkte wie $K$ verbindet. Dann besagt unser Satz, dass $G^*$ eine Euler-Linie besitzt. Wir müssen daher beweisen, dass $G^*$

[36] Man versteht unter einem von beiden Seiten unendlichen Kantenzug eine Folge der Kanten

$$\cdots, P_{-1}P_0, P_0P_1, \cdots$$

in der eine jede Kante höchstens einmal vorkommt. ähnlicher Weise ist ein einseitig unendlicher Kantenzug die Folge der Kanten

$$P_0P_1, P_1P_2, \cdots$$

die jede Kante höchstens einmal enthält.

[37] Daraus folgt auch, dass $G$ weniger als $C$ unendliche Glieder hat.

[38] $E$ könnte man auch hier durch $E'$ ersetzen.

die Bedingungen $T_1$, $T_2$, $T_3$, $E_1$ und $E_2$ dann und nur dann erfüllt, wenn $G$ den Bedingungen $T_1$, $T_2$ und $E$ genüge leistet.

Es ist unmittelbar klar, dass $G^*$ die Bedingungen $T_1$ und $T_2$ dann und nur dann erfüllen kann, wenn auch $G$ dieselben Bedingungen erfüllt. Ferner entspricht $G^*$ der Bedingung $T_3$ immer, da er doch durch die Verdoppelung der Kanten von $G$ entstanden ist. Wir haben daher nur zu zeigen, dass $G^*$ die Bedingung $E_1$ und $E_2$ dann und nur dann erfüllt, wenn $G$ der Bedingung $E$ entspricht.

Wir setzen zuerst voraus, dass $G$ die Bedingung $E$ nicht erfüllt, d. h. es existiert ein endlicher Graph $g \subset G$ für welchen der Graph $G - g$ mehrere unendliche Komponenten hat und beweisen sofort, dass in diesem Falle $G^*$ die Bedingung $E_2$ nicht erfüllt. Wir bezeichnen nämlich die unendlichen Komponenten des Graphen $G - g$ durch $\gamma_1, \gamma_2, \cdots$. Wir verdoppeln die Kanten von $g, \gamma_1, \gamma_2, \cdots$ und erhalten dadurch die Graphen $g^*, \gamma_1^*, \gamma_2^*, \cdots$. Dann sind die unendlichen Komponenten des Graphen $G^* - g^*$ eben $\gamma_1^*, \gamma_2^*, \cdots$. Da aber $g^*$ ein Graph geraden Grades ist bedeutet dies eben, dass der Graph $G^*$ Bedingung $E_2$ nicht erfüllt.

Jetzt wollen wir zeigen, dass wenn $G$ die Bedingung $E$ erfüllt, so erfüllt $G^*$ nicht nur $E_1$ und $E_2$ sondern auch $E$. Denn, dass $G^*$ der Bedingung $E$ nicht entspricht, würde bedeuten, dass es einen endlichen Graph $\mathfrak{g} \subset G^*$ gibt, für den der Graph $G^* - \mathfrak{g}$ mehrere unendliche Komponenten hat. Betrachten wir nun die Kante $K \subset \mathfrak{g}$. Aus der Entstehung von $G^*$ ist es klar, dass es auch eine andere Kante $K' \subset G^*$ gibt, welche die selben 2 Knotenpunkte, wie $K$ verbindet. Wenn $K' \subset \mathfrak{g}$, so adjungieren wir sie zu $\mathfrak{g}$; wir erhalten so einen endlichen Graph geraden Grades $\mathfrak{g}^*$. Da der Graph $G^* - \mathfrak{g}$ nach der Voraussetzung mehrere unendliche Komponenten besitzt und da, wegen $\mathfrak{g} \subset \mathfrak{g}^*$, $G^* - \mathfrak{g} \supset G^* - \mathfrak{g}^*$, so hat auch $G^* - \mathfrak{g}^*$ mehrere unendliche Komponenten $\bar{\gamma}_1^*, \bar{\gamma}_2^*, \cdots$. Wir bezeichnen nun die aus den in $G$ liegenden Kanten der Graphen $\mathfrak{g}^*, \bar{\gamma}_1^*, \bar{\gamma}_2^*, \cdots$ gebildete Graphen durch $g, \gamma_1, \gamma_2, \cdots$ (durch Verdoppelung der Kanten dieser Graphen ergeben sich die Graphen $\mathfrak{g}^*, \bar{\gamma}_1^*, \bar{\gamma}_2^*, \cdots$). Die Graphen $\gamma_1, \gamma_2, \cdots$ sind die unendlichen Komponente des Graphen $G-g$. Da aber $g$ ein endlicher Graph ist, steht dieses Ergebnis mit der Voraussetzung dass $G, E$ erfüllt in Widerspruch. Damit haben wir also bewiesen, dass wenn $G^*$ die Bedingung $E$ nicht erfüllt, so erfüllt sie auch $G$ nicht. Also wenn $G$ durch $E$ erfüllt ist, so wird sie auch durch $G^*$ erfüllt.

Womit unsere Behauptung vollständig bewiesen ist.

5,3. Wir verbinden den Gitterpunkt $(x_1, x_2, \cdots, x_n)$ des $n$ dimensionalen Raumes ($x$ beliebige ganze Zahlen) durch Kanten mit den Gitterpunkten $(x_1 + 1, x_2, \cdots, x_n)$, $(x_1, x_2 + 1, \cdots x_n)$, $\cdots$ $(x_1, x_2, \cdots, x_n + 1)$. Wenn wir diese Verbindungen für alle Gitterpunkte des Raumes durchgeführt haben, erhalten wir den gewöhnlichen Gittergraph des $n$ dimensionalen Raumes.

Es fragt sich, ob der gewöhnliche Gittergraph des $n$ dimensionalen Raumes eine Euler-Linie besitzt.

Aus den Resultaten unserer Arbeit ergibt sich die bejahende Antwort sehr einfach.[19] Es ist nämlich evident, dass der $n$ dimensionale Gittergraph $T_1$ und $T_2$ erfüllt, er erfüllt auch $T_3$, da sich in einem jeden Knotenpunkt $2n$ Kanten treffen. Es wird ferner auch $E$ erfüllt, denn wenn wir aus dem Graph einen beliebigen endlichen Graph weglassen, hat der zurückbleibende Graph nur ein einziges unendliches Glied.

Es gilt also folgender Satz: *der gewöhnliche Gittergraph des n dimensionalen Raumes besitzt eine Euler-Linie.*

BUDAPEST, HUNGARY.

[19] Einen direkten Beweis dieses Satzes gibt die demnächst in der Acta Lentei. ca Litt Szeged erscheinende Arbeit von E. Vázsonyi: Über Gitterpunkte des mehrdimensionalen Raumes.

# PERFECT SUBDIVISION OF SURFACES

By I. N. Kagno*

*Introduction.* In a previous paper[1] we derived theorems on the triangulation of surfaces by graphs. In the present paper we further study the separation of surfaces by graphs.

Consider the question: If a connected graph $G$, containing no circuits of less than three arcs (i.e. no simple loops as defined in $T$), and no vertices of degree less than three, triangulates an orientable (non-orientable) surface of characteristic $K$, can it be mapped on a non-orientable (orientable) surface of the same characteristic? This problem appears to be difficult.[2] In the present paper we attempt a partial answer to the question. We introduce the concept of the 1-star of a vertex of a graph. If the structure of $G$ is such that every vertex of $G$ has a unique 1-star, then the question can be answered in the negative (theorem 4). Thus for graphs having this structure and the additional property that they triangulate their maximal surface, there is defined a sort of orientability or non-orientability relative to their maximal surface. The question can also be answered in the negative if the 1-stars of $G$ have a certain structure (theorem 7). We further study the 1-stars of graphs which triangulate surfaces, and show that if $G$ triangulates a surface, an upper bound to the number of 1-stars a vertex of $G$ can have is expressible in terms of the characteristic of the surface.

We study also the perfect subdivision of surfaces, where by perfect subdivision of a surface we mean the separation of the surface by a graph into regular 2-cells of $n$ sides each satisfying certain conditions as set forth in section 2. In the special case where $n = 3$, this sub-

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[1] "The triangulation of surfaces and the Heawood color formula"—Journal of Math. and Physics, Vol. XV (1936), pp. 179–186. We shall refer to this paper as T.

[2] In the paper T we stated that when $G$ is a complete $n$-point (i.e. a graph formed by joining each of $n$ vertices to each of the others) the answer is "no," but since the appearance of the paper T it has come to the author's attention that his proof of this statement is erroneous.

division is a triangulation. We derive necessary conditions for perfect subdivision. We define a perfect polyhedron as the 2-complex formed by the perfect subdivision of a surface by a regular graph, and derive necessary conditions for the existence of such polyhedra. In particular we show that the surfaces of the five Platonic solids are the only perfect polyhedra arising from the perfect subdivision of a sphere, and give a non-metrical proof of the theorem of Euclidean Geometry which states that there are only five regular solids.

In the last section we define and study fundamental domains of the universal covering surfaces of orientable surfaces and derive a perfect subdivision of these covering surfaces. We apply the results to show that the fundamental domain of a doubly periodic function of a complex variable is either a hexagon or a quadrilateral.

The definitions and notations of the paper $T$ shall apply throughout the present paper. In the sequel, unless otherwise indicated, let $G$ be a finite connected graph containing no simple loops and no vertices of degree less than three.

**1. 1-stars of a graph. Theorems on Triangulation.**

*Theorem* 1. *Suppose $G$ triangulates a surface $\mathfrak{S}_K$. Let $G$ be mapped on any other surface $\mathfrak{S}'_K$ of the same characteristic. Then $G$ triangulates $\mathfrak{S}'_K$ also.*

Proof: Suppose $G$ does not triangulate $\mathfrak{S}'_K$. We shall force a contradiction. $\mathfrak{S}'_K$ is maximal for $G$, since $G$ triangulates $\mathfrak{S}_K$, and a graph which triangulates a surface of characteristic $K$ cannot be mapped on a surface of greater characteristic.[3] A necessary and sufficient condition that $G$ be mapped on its maximal surface is that the surface be separated into 2-cells by $G$,[4] hence $\mathfrak{S}'_K$ is separated into 2-cells. Let $a_i$ be the number of vertices of degree $i$ in $G$, $(i = 3, 4, \cdots, \alpha_0 - 1)$. If a graph is mapped on a surface which it triangulates then the number of 2-cells is given by $\dfrac{\sum_{i=3}^{\alpha_0-1} ia_i}{3}$.[5] If a graph is mapped on a surface which

[3] T. Theorem 3.

[4] T. Theorem 2. The necessity of this condition for *orientable* surfaces was known to D. Konig before the appearance of the paper T. In his book "Theorie der Endlichen und Unendlichen Graphen," p. 198 he states, "The following theorem is easily shown: If $F$ is a closed orientable surface of least possible genus on which a given connected graph can be mapped then it separates this surface into surface elements (Elementarflächen)."

[5] T. Lemma 1.

it does not triangulate, but which it separates into 2-cells, then the number of 2-cells is less than $\frac{\sum ia_i}{3}$.[6] If $\alpha_2$ is the number of 2-cells on $\mathfrak{S}_K$ and $\alpha_2'$ the number of 2-cells on $\mathfrak{S}_K'$, then $\alpha_2 = \frac{\sum ia_i}{3}$, $\alpha_2' < \frac{\sum ia_i}{3}$. But since $K$ is the same for both surfaces we must have $\alpha_2 = \alpha_2'$.

Suppose a graph $G$ is mapped on a surface $\mathfrak{S}$. Let $v$ be any vertex of $G$. By the *2-star* of $v$ we mean the 2-complex consisting of $v$ and the 1-cells and portions of $\mathfrak{S} - G$ incident with $v$.

*Theorem* 2. *Suppose $G$ triangulates a surface $\mathfrak{S}$. Let $v$ be any vertex of $G$. Then the boundary of the 2-star of $v$ is a simple closed curve, (i.e. a circuit of $G$).*

Proof: Each 2-cell incident with $v$ abuts at $v$ only once, for otherwise, since this 2-cell is triangular, some arc of its boundary would have $v$ for both its endpoints and would thus be a simple loop. But this is impossible since $G$ contains no simple loops. Hence $\sigma_v$, the 2-star of $v$, consists of a number of arcs and triangular 2-cells, each incident with $v$. $\lambda$, the boundary of $\sigma_v$, consists of those arcs of the boundaries of these 2-cells which are not incident with $v$. $\lambda$ is a simple closed curve, for otherwise $\lambda$ would meet itself in some vertex $u$, but then there would be two arcs joining $u$ to $v$, i.e. $G$ would contain a simple loop.

Suppose a graph $G$ contains among other, the vertices $(a, b, \cdots, c)$, and that in $G$ there are several circuits containing each of these vertices. We say one such circuit $\lambda$ is *smaller* than another such circuit $\mu$ if $\lambda$ contains fewer arcs than $\mu$.[7]

Two vertices of $G$ shall be called *adjacent* if they are joined by an arc in $G$. Let $v$ be a vertex of $G$, and $v_1, \cdots, v_n$ be the set of vertices of $G$ which are adjacent to $v$. If $G$ contains a circuit passing through each of the vertices $v_1, \cdots, v_n$, then of all such circuits there will be one or more, $\lambda_1, \cdots, \lambda_m$ which are smallest. By a *1-star* of $v$ we mean the subgraph consisting of $v$, the arcs $vv_i$, $(i = 1, \cdots, n)$ and one of the circuits $\lambda_j$, $(j = 1, \cdots, m)$. The circuit $\lambda_j$ shall be called the *rim* of the 1-star. Obviously $v$ need not necessarily have any 1-stars.

*Theorem* 3. *Suppose $G$ triangulates some surface $\mathfrak{S}$. Let $v$ be any*

[6] T. Lemma 2.

[7] Example: Suppose $\lambda = 1231$, $\mu = 143251$. Each of these circuits contains the vertices 1, 2, 3, but $\lambda$ contains three arcs while $\mu$ contains five. $\lambda$ is smaller than $\mu$.

*vertex of G. Then v has at least one* 1*-star, and the rim of any* 1*-star of v contains only vertices adjacent to v.*

Proof: Consider the 2-star of $v$ on $\mathfrak{S}$. Its boundary is a circuit of $G$, and since the surface is triangulated, $v$ is joined to every vertex on this circuit, and every vertex adjacent to $v$ is on this circuit. This circuit is thus the smallest which contains the vertices adjacent to $v$, and therefore is the rim of a 1-star of $v$. Since the rim of any other 1-star of $v$ is a circuit not larger than this one, the theorem follows.

*Corollary. Suppose G triangulates some surface* $\mathfrak{S}$. *If v is any vertex of G the boundary of the* 2*-star of v is also the rim of a* 1*-star of v.*

*Theorem* 4. *Suppose each vertex of G has exactly one* 1*-star, and that G triangulates an orientable (non-orientable) surface* $\mathfrak{S}_K$ *of characteristic K. Then G cannot be mapped on a non-orientable (orientable) surface of characteristic K.*

Proof: Let $v$ be any vertex of $G$. The boundary of the 2-star of $v$ on $\mathfrak{S}_K$ is a simple closed curve $\lambda$. By the corollary to theorem 3, $\lambda$ is also the rim of the unique 1-star of $v$.

Suppose $G$ be mapped on any other surface $\mathfrak{S}'_K$ of the same characteristic. By theorem 1, $G$ also triangulates $\mathfrak{S}'_K$. Let $\sigma_v$ be the 2-star of $v$ on $\mathfrak{S}_k$, $\sigma'_v$ the 2-star of $v$ on $\mathfrak{S}'_K$. $F(\sigma'_v)$, the subgraph of $G$ which forms the boundary of $\sigma'_v$, is a simple closed curve, which must also be the rim of the unique 1-star of $v$, hence is identical with $\lambda$.

Let $a$, $b$, be any two adjacent vertices of $F(\sigma_v)$. Then since $F(\sigma'_v) \equiv \lambda \equiv F(\sigma_v)$, $a$, $b$ are also adjacent vertices of $F(\sigma'_v)$. On $\mathfrak{S}_K$ the circuit *avba* bounds a triangle of $\sigma_v$. On $\mathfrak{S}'_K$ the circuit *avba* bounds a triangle of $\sigma'_v$. Since $v$ is an arbitrary vertex of $G$, it follows that if any circuit *xyzx* of $G$ bounds a triangle on $\mathfrak{S}_K$, (this triangle is part of the 2-star of the vertex $x$) *xyzx* also bounds a triangle on $\mathfrak{S}'_K$. Similarly we can show that any circuit of $G$ which bounds a triangle on $\mathfrak{S}'_K$ also bounds a triangle on $\mathfrak{S}_K$. Hence we can set up a $1-1$, incidence preserving, correspondence between the 0-, 1-, and 2-cells of $\mathfrak{S}'_K$ and the 0-, 1-, and 2-cells of $\mathfrak{S}_K$, i.e. the two surfaces are isomorphic, and $\mathfrak{S}'_K$ is orientable or non-orientable as $\mathfrak{S}_K$ is.

*Example*: As evidence that a graph of the type discussed in the theorem actually exists, consider the map of a graph which triangulate a torus given in Fig. 1. (In this figure, and in those that follow, the free edges are identified as indicated.) The unique 1-star of the vertex 8 consists of the vertex 8, the arcs $\widehat{82}$, $\widehat{83}$, $\widehat{89}$, $\widehat{8\,11}$, $\widehat{86}$, $\widehat{85}$, and the circuit 2 3 9 11 6 5 2. Similarly for the other vertices. By the theorem, this graph cannot be mapped on a non-orientable surface of characteristic 0.

We digress at this point to consider a problem on Hamilton lines. Let $\lambda$ be a circuit, of a graph $G$, containing every vertex of $G$, (i.e. a closed path which passes through every vertex once and only once), if such exists. $\lambda$ is called a *Hamilton line*. The problem we consider is: Let $G$ contain $n$ vertices, each of degree $\leq t$. What is an upper bound to the number of Hamilton lines which can be traced on $G$, for all possible choices of $G$?

Let $\sigma$ be this upper bound. Let us start at an arbitrary, but fixed, vertex $v$ of $G$, and proceed to trace a Hamilton line $\lambda$. There are at most $t$ possible ways of choosing the first arc $\widehat{vu}$ of $\lambda$, since $v$ is of degree $\leq t$. There are at most $(t-1)$ ways of choosing the second arc $\widehat{uw}$

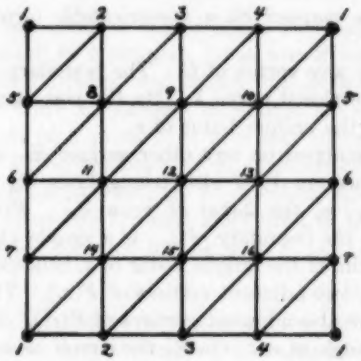


FIG. 1

of $\lambda$, since $u$ is of degree $\leq t$. Similarly there are at most $(t-1)$ ways of choosing the $j$th arc of $\lambda$, $(j \leq n-t+1)$. After we have choosen the $(n-t+1)$th arc, $\lambda$ shall have passed through $(n-t+1)$ vertices, the free endpoint of the $(n-t+1)$th arc being the $(n-t+2)$nd vertex through which $\lambda$ will pass. That is, there are left $t-2$ vertices through which $\lambda$ has yet to pass. In other words, there are at most $t-2$ possible ways of choosing the $(n-t+2)$nd arc of $\lambda$. Similarly there are left $t-3$ vertices through which $\lambda$ has yet to pass, after the choice of the $(n-t+2)$nd arc of $\lambda$. That is there are left $t-3$ possible ways of choosing the $(n-t+3)$rd arc of $\lambda$. Continuing in this manner, we have finally, there are two ways of choosing the $(n-2)$nd arc, one way of choosing the $(n-1)$st arc, and one way of

choosing the last arc. That is, the number of ways of choosing $\lambda$ is at most

$$s = t(t-1)^{(n-t+1)-1}\cdot(t-2)!.$$

But in this enumeration, each Hamilton line has been counted twice, once in the positive sense and once in the negative sense. Hence,

$$\sigma = \frac{s}{2} = \frac{t!(t-1)^{n-t-1}}{2}.$$

*Lemma* 1. *Let $G$ be a graph containing $n$ vertices, each of degree $\leq t$. Then the number of Hamilton lines on $G$ is at most* $\dfrac{t!(t-1)^{n-t-1}}{2}$.

*Corollary. Let $G$ be a graph containing $n$ vertices. Then the number of Hamilton lines on $G$ cannot exceed* $\dfrac{(n-1)!}{2}$.

*Theorem* 5. *Let $v$ be any vertex of $G$, $d_v$ the degree of $v$. Then if $G$ triangulates a surface $\mathfrak{S}$, the number of* 1*-stars of $v$ cannot exceed* $\dfrac{(d_v-1)!}{2}$, *and the total number of* 1*-stars in $G$ cannot exceed* $\sum_{i=1}^{\alpha_0}\dfrac{(d_{v_i}-1)!}{2}$.

Proof: Consider the 2-star of $v$ on $\mathfrak{S}$. Its boundary $\lambda$ is a circuit which contains all the vertices adjacent to $v$, and no others, and $\lambda$ is also the rim of a 1-star of $v$. Let $H$ be the subgraph consisting of the vertices adjacent to $v$ and all arcs which join pairs of these vertices. The rim of any 1-star of $v$ is a Hamilton line of $H$. Since $H$ has $d_v$ vertices, the number of such Hamilton lines cannot exceed $\dfrac{(d_v-1)!}{2}$.

We shall say two 1-stars of $G$ are *adjacent* if their centers are adjacent vertices of $G$. ($v$ is called the *center* of any 1-star of $v$.)

*Theorem* 6. *Let $G$ be such that the rims of any two adjacent* 1*-stars of $G$ have at most $t$ vertices in common. Let $v$ be any vertex of $G$, $d_v$ the degree of $v$. Then if $G$ triangulates a surface $\mathfrak{S}$, $v$ has at most* $\dfrac{t!(t-1)^{d_v-t-1}}{2}$ *distinct* 1*-stars.*

Proof: Let $v$ be any vertex of $G$, $d_v$ its degree. Consider the 2-star of $v$ on $\mathfrak{S}$. Its boundary $\lambda$ is a circuit which contains all the vertices adjacent to $v$, and no others, and $\lambda$ is also the rim of a 1-star of $v$. Suppose $\mu$ is the rim of another 1-star of $v$, if such exists. $\mu$ cannot be

larger nor smaller than $\lambda$, by the definition of a 1-star. That is, the rim of any 1-star of $v$ will contain all the vertices of $\lambda$ and no others.

Let $v'$ be adjacent to $v$. Then $v'$ is contained in $\lambda$. The rim of a 1-star of $v'$ has at most $t$ vertices in common with $\lambda$ by hypothesis. That is, any vertex of $\lambda$ is adjacent to at most $t$ other vertices of $\lambda$.

Let $H$ be the subgraph of $G$ consisting of the vertices of $\lambda$, and all arcs of $G$ both of whose endpoints are vertices of $\lambda$. Each vertex of $H$ is of degree $\leq t$ on $H$. The rim of any 1-star of $v$ is a Hamilton line of $H$. By lemma 1 there are at most $\frac{t!(t-1)^{d_v-t-1}}{2}$ such Hamilton lines.

*Theorem* 7. *Let $G$ be such that the rims of any two adjacent* 1*-stars of $G$ have exactly two vertices in common. Then if $G$ triangulates an orientable (non-orientable) surface $\mathfrak{S}_K$ of characteristic $K$, $G$ cannot be mapped on a non-orientable (orientable) surface of the same characteristic.*

Proof: If $t = 2$ in theorem 6, we find that each vertex of $G$ has a unique 1-star. The theorem follows from this and theorem 4.

Empirical tests with many graphs have led the author to the belief that the following assertion is true: "Suppose any connected graph, containing no vertices of degree less than three, and no loops, triangulates an orientable (non-orientable) surface of characteristic $K$. Then $G$ cannot be mapped on a non-orientable (orientable) surface of the same characteristic." As we have pointed out in the Introduction, the proof of this assertion, if it is true, appears to be difficult, and we leave it as an open question.

*Lemma* 2. *Let $\alpha$ be a* 2*-cell bounded by a simple closed curve $\lambda$ having $n$ edges and $n$ vertices. Let $[\alpha]$ be a non-intersecting set of arcs, each of which, except for its endpoints, is mapped in $\alpha$, such that* (1) *the endpoints of these arcs are non-adjacent vertices of $\lambda$, and* (2) *no pair of vertices of $\lambda$ has more than one arc of $[\alpha]$ joining them,* (*i.e. the graph $\lambda + [\alpha]$ contains no simple loops.*) *Then the number of arcs in $[\alpha]$ cannot exceed $n - 3$.*

Proof: For $n \leq 3$ the lemma is trivial. When $n = 4$ it is obvious that $[\alpha]$ cannot contain more than one arc.

Suppose the lemma is true for $n = k$, we shall show that it is true for $n = k + 1$. Let $uv$ be any arc of $[\alpha]$. $uv$ separates $\alpha$ into two 2-cells $\alpha'$ and $\alpha''$. $F(\alpha') = \lambda' + uv$, $F(\alpha'') = \lambda'' + uv$, where $\lambda'$, $\lambda''$ are the two chains into which $u$, $v$ separate $\lambda$. Since by hypothesis $u$ and $v$ are non-adjacent vertices of $\lambda$, $\lambda'$ contains at least two arcs of $\lambda$, so that $\lambda''$ contains at most $k + 1 - 2$ $(= k - 1)$ arcs. Similarly $\lambda'$ contains at most $k - 1$ arcs. Let $m$ be the number of arcs in $\lambda'$. Then

$\lambda''$ contains $k + 1 - m$ arcs, $(m \leq k - 1, k + 1 - m \leq k - 1)$. Then $F(\alpha')$ contains $m + 1$ arcs, and since $m + 1 \leq k$, it follows that the number of arcs of $[\alpha]$ mapped in $\alpha'$ cannot exceed $(m + 1) - 3$. Similarly $F(\alpha'')$ contains $k + 2 - m$ arcs, and since $k + 2 - m \leq k$, the number of arcs of $[\alpha]$ mapped in $\alpha''$ cannot exceed $(k + 2 - m) - 3$. Taking account of the arc $uv$, the number of arcs of $[\alpha]$ mapped in $\alpha$ cannot exceed

$$[(m + 1) - 3] + [(k + 2 - m) - 3] + 1 = (k + 1) - 3$$

and hence the lemma is true for $n = k + 1$.

*Lemma* 3. *Let $\beta$ be a 2-cell with $p$ handles, bounded by a simple closed curve $\mu$ having $n$ edges and $n$ vertices. Let $[\beta]$ be a non-intersecting set of arcs, each of which, except for its endpoints, is mapped in $\beta$, having the same properties relative to $\beta$ and $\mu$ that $[\alpha]$ has relative to $\alpha$ and $\lambda$ in lemma* 2. *Then the number of arcs in $[\beta]$ cannot exceed $n - 3K + 3$, where $K = 2 - 2p$.*

Proof: *Case I.* Suppose each handle has at least one of the arcs of $[\beta]$ passing over it, and at least one of the arcs of $[\beta]$ passing under it. Let $\xi_i$ be an arc which passes over the $i$th handle, $\eta_i$ be an arc which passes under the $i$th handle, $(i = 1, \cdots, p)$. Cut $\beta$ along each of the $\xi_i$ and $\eta_i$. $\beta$ will open out into a 2-cell $\beta^*$ bounded by the $n$ edges of $\mu$ together with $2p$ identified pairs of arcs $\xi_i$, $\xi_i'$, $\eta_i$, $\eta_i'$, that is, by $n + 4p$ edges, and the other arcs of $[\beta]$ will all be mapped in $\beta^*$.

$\beta^*$ and those arcs of $[\beta]$ which are mapped in $\beta^*$ satisfy the hypothesis of lemma 2, hence the number of arcs of $[\beta]$ mapped in $\beta^*$ cannot exceed $n + 4p - 3$. These arcs, together with the $2p$ arcs $\xi_i$, $\eta_i$, give a total not exceeding $n + 6p - 3$ for the number of arcs in $[\beta]$. Putting $K = 2 - 2p$ we find that the number of arcs in $[\beta]$ cannot exceed $n - 3K + 3$.

*Case II.* Suppose not every handle has at least one arc of $[\beta]$ passing over it, and at least one arc of $[\beta]$ passing under it. Suppose $h_1$ of the handles do not have any arcs of $[\beta]$ passing over them. Pick off[4] all such handles. There results a 2-cell $\beta'$ with $p - h_1$ handles, each of which has at least one arc of $[\beta]$ passing over it. Suppose $h_2$ of these handles do not have any arcs of $[\beta]$ passing under them. Then we can transform the 2-complex $\mu + \beta' + [\beta]$ into a homeomorphic 2-complex $\mu + \beta'' + [\beta]$ such that each handle of $\beta'$ which has arcs passing over it,

[4] If we delete a handle from $\beta$ there results a region with two holes. Span each of these holes with a 2-cell. There will result a 2-cell $\beta'$ with one fewer handles. This process is called picking off a handle.

but none passing under it, is replaced on $\beta''$ by a handle which has these arcs passing under it but none passing over it. Pick off all such handles. There will result a 2-cell $\beta'''$ with $p - h_1 - h_2$ handles, in which the arcs of $[\beta]$ are mapped. Each of these handles will have at least one arc of $[\beta]$ passing over it, and at least one arc of $[\beta]$ passing under it. We can then apply case I to $\beta'''$, and find that the number of arcs in $[\beta]$ cannot exceed $\tau = n + 6(p - h) - 3$, where $h = h_1 + h_2$. By the hypothesis of case II, $h \geq 1$. Putting $K = 2 - 2p$ we see that $\tau = (n - 3K + 3) - 6h < n - 3K + 3$.

*Lemma* 4. *Let $\gamma$ be a 2-cell with $q$ cross-caps, bounded by a simple closed curve $\nu$ having $n$ edges and $n$ vertices. Let $[\gamma]$ be a non-intersecting set of arcs, each of which, except for its endpoints, is mapped in $\gamma$, having the same properties relative to $\gamma$ and $\nu$ that $[\alpha]$ has relative to $\alpha$ and $\lambda$ in lemma* 2. *Then the number of arcs in $[\gamma]$ cannot exceed $n - 3K + 3$, where $K = 2 - q$.*

The proof, with slight modifications, is analogous to that of the lemma 3, "cross-caps" replacing "handles" in the discussion.

*Lemma* 5. *Let $H$ be a graph containing $E$ arcs and $V$ vertices. Then the number of Hamilton lines on $H$ is less than*

$$\frac{E(E-1)\cdots(E-V+1)}{V!}.$$

Proof: Each Hamilton line is a circuit of $V$ arcs. $\omega$, the number of Hamilton lines, cannot exceed the number of ways of choosing $V$ arcs from $E$, i.e. $\omega \leq {}^{E}C_{V}$. But not every such choice of $V$ arcs is a circuit, hence the equality cannot hold and

$$\omega < \frac{E(E-1)\cdots(E-V+1)}{V!}.$$

*Theorem* 8. *If $G$ triangulates an $\mathfrak{S}_K$, and $v$ is any vertex of $G$, $d_v$ the degree of $v$, then $\dfrac{(2d_v - 3K + 3)(2d_v - 3K + 2)\cdots(d_v - 3K + 4)}{d_v!}$ is an upper bound to the number of distinct 1-stars of $v$.*

Proof: If $\mathfrak{S}_K$ is orientable, $G$ can be mapped on a sphere $S_K$ with $p = (2 - K)/2$ handles. If $\mathfrak{S}_K$ is non-orientable then $G$ can be mapped on a sphere $S_K$ with $q = (2 - K)$ cross-caps. Whichever is the case denote the surface by $S$, and map $G$ on it. $G$ triangulates $S$, (Theorem 1). Consider $\sigma_v$, the 2-star of $v$ on $S$. Its boundary is a simple closed curve $\lambda$, (Theorem 2) containing $d_v$ arcs and $d_v$ vertices. $\lambda$ is the rim of a 1-star of $v$, (Theorem 3, corollary). If $\mu$ is the rim of any other

1-star of $v$, $\mu$ obviously contains at least one arc not contained in $\lambda$. Thus for each 1-star of $v$ distinct from the one whose rim is $\lambda$, there exists one or more arcs mapped in $(S - \bar{\sigma}_v)$, the portion of $S$ complementary to the closure of $\sigma_v$, these arcs having the properties that the arcs in the sets $[\beta]$ and $[\gamma]$ of lemmas 3 and 4 have. Denote the set of these arcs by $[\delta]$. Let $\tau$ be the number of arcs in $[\delta]$. Then by lemma 3 or lemma 4, $\tau \leq d_v - 3K + 3$.

Let $H$ be the subgraph of $G$ consisting of $\lambda$ and the set $[\delta]$. Then every rim of a 1-star of $v$ is a Hamilton line of $H$. Since $H$ has $d_v + \tau$ arcs and $d_v$ vertices, by lemma 5 the number of such Hamilton lines is less than

$$\frac{(d_v + \tau)(d_v + \tau - 1) \cdots (\tau + 1)}{d_v!}.$$

That is, the number of distinct 1-stars of $v$ cannot exceed

$$\frac{(2d_v - 3K + 3)(2d_v - 3K + 2) \cdots (d_v - 3K + 4)}{d_v!}.$$

We emphasize the point that the bound given in theorem 8 is by no means the least upper bound. In actual practice the number of distinct 1-stars of $v$ may be much less than the number given in the theorem. For example, in the case of the plane $K = 2$, and the upper bound given is $\dfrac{(2d_v - 3)(2d_v - 4) \cdots (d_v - 2)}{d_v!}$. However for the plane we have the

*Theorem* 9. *If $G$ triangulates a plane, then every vertex of $G$ has a unique* 1*-star.*

Proof: Since $G$ is planar, it can be mapped on a sphere. Let $v$ be any vertex of $G$. Consider the 2-star of $v$. Its boundary $\lambda$ is the rim of a 1-star of $v$ (theorem 3, corollary). If $v$ has another 1-star, let $\mu$ be its rim. At least one arc $xy$ of $\mu$ is not contained in $\lambda$, else $\mu$ and $\lambda$ would be identical. Suppose $xy$ is the $i$th arc of $\mu$. The pair of vertices $x$, $y$ separate $\lambda$ into two chains $\lambda_1$ and $\lambda_2$. If the $(i + 1)$st arc of $\mu$ joins $y$ to a vertex of $\lambda_1$ then since $\mu$ passes through every vertex of $\lambda_2$, $\mu$ must contain an arc $uw$ joining a vertex $u$ of $\lambda_1$ to a vertex $w$ of $\lambda_2$. Similarly if the $(i + 1)$st arc of $\mu$ joins $y$ to a vertex of $\lambda_2$. The pair $x$, $y$ separates the pair $u$, $w$ on $\lambda$. Now $uw$ and $xy$ must both be mapped in the region complementary to the 2-star of $v$. But since $x$, $y$ separates $u$, $v$ this is impossible. Hence $v$ has a unique 1-star.

Two circuits of a graph are called *independent* if they have no points other than vertices of the graph in common. Suppose $v$ is a vertex of $G$, and that $v$ has 1-stars. We shall say two 1-stars of $v$ are *independent* if their rims are independent circuits of $G$.

*Theorem* 10. *Let $v$ be any vertex of $G$, $d_v$ the degree of $v$. If $G$ triangulates some surface $\mathfrak{S}$ then the number of independent* 1*-stars of $v$ cannot exceed* $\dfrac{(d_v - 1)(d_v - 2)}{2}$.

Proof: Consider the 2-star of $v$ on $\mathfrak{S}$. Its boundary $\lambda$ is the rim of a 1-star of $v$. Let $H$ be the subgraph of $G$ consisting of $\lambda$ and those arcs of $G$ both of whose endpoints are vertices of $\lambda$. The rim of any 1-star of $v$ is a Hamilton line of $H$, i.e. a circuit of $H$. The number of independent circuits of $H$ cannot exceed $\mu$, the cyclomatic number of $H$. $\mu$ will have the greatest possible value when $H$ is a complete $d_v$-point. The number of independent circuits of such a graph is

cyclomatic

$$\alpha_1 - \alpha_0 + 1 = \frac{d_v(d_v - 1)}{2} - d_v + 1$$

$$= \frac{(d_v - 1)(d_v - 2)}{2}.$$

Hence in $H$ $\mu \leq \dfrac{(d_v - 1)(d_v - 2)}{2}$ and the number of independent 1-stars of $v$ cannot exceed this amount.

*Theorem* 11. *If $G$ triangulates an $\mathfrak{S}_K$, and $v$ is any vertex of $G$, $d_v$ the degree of $v$, then the number of independent* 1*-stars of $v$ cannot exceed* $d_v - 3K + 4$.

Proof: $G$ can be mapped on a sphere $S_K$ with $p = (2 - K)/2$ handles or with $q = (2 - K)$ cross-caps, where $S_K$ is defined as in the proof of theorem 8. Let $\tau$ be defined as in theorem 8, and $H$ be the subgraph of $G$ as there defined. The rim of any 1-star of $v$ is a Hamilton line of $H$. Since $H$ has $d_v$ vertices and $d_v + \tau$ arcs, the number of independent circuits in $H$ is given by

cyclomatic

$$\mu = \alpha_1 - \alpha_0 + 1$$

$$= (d_v + \tau) - d_v + 1$$

$$= \tau + 1.$$

As we have seen in the proof of theorem 8, $\tau \leq d_v - 3K + 3$. Hence the number of independent Hamilton lines in $H$ (which cannot exceed

$\mu$), is $\leq d_v - 3K + 4$, and this number is therefore an upper bound to the number of independent 1-stars of $v$.

2. **Perfect and Quasi-perfect subdivision.** In the sequel let $H$ be any connected finite graph, each vertex of which is of degree at least three, and let $G$ be defined as before. $H$ shall be said to subdivide a surface $\mathfrak{S}$, on which it is mapped, *quasi-perfectly* if the regions into which $\mathfrak{S}$ is separated are 2-cells each having the same finite number $n$ of sides.[9] $H$ shall be said to subdivide $\mathfrak{S}$ *perfectly* if the regions into which $\mathfrak{S}$ is separated are regular[10] 2-cells having the same finite number $n$ of sides, and if the boundaries of any two of these 2-cells meet in at most a connected set. If $H$ subdivides $\mathfrak{S}$ perfectly, it also subdivides $\mathfrak{S}$ quasi-perfectly, so that the following theorems on quasi-perfect subdivisions are also true for perfect subdivisions.

*Theorem* 12. *Suppose $H$ can be mapped on a surface so as to subdivide it quasi-perfectly. Then if $n$ denote the number of sides of each 2-cell,* $n = \dfrac{2\alpha_1}{\alpha_2}$.

Proof: Let $a_i$ be the number of vertices of degree $i$ in $H$. At each vertex of $H$ there are $i$ abutments of 2-cells. (A given 2-cell may abut more than once at a given vertex of $H$.) Hence the total number of abutments at all vertices is $\sum_{i=3}^{\alpha_0-1} ia_i$. Each 2-cell abuts $n$ times at vertices of $H$. Hence in the above enumeration each 2-cell has been counted $n$ times, once each time it abuts at a vertex of $H$. Hence

$$\alpha_2 = \frac{\sum ia_i}{n} \tag{1}$$

But $\Sigma ia_i$ is also equal to the number of abutments of arcs at vertices, and since each arc abuts twice, once at each of its ends, $\sum ia_i = 2\alpha_1$. This combined with (1) gives $n = \dfrac{2\alpha_1}{\alpha_2}$.

*Theorem* 13. *If for a graph $H$, $\dfrac{2\alpha_1}{\alpha_1 - \alpha_0 + K}$ is not an integer, then $H$ cannot be mapped on an $\mathfrak{S}_K$ so as to subdivide it quasi-perfectly.*

[9] Defined in T as follows: Let $F(\alpha)$ be the boundary of a 2-cell $\alpha$. Let $P$ be an arbitrary but fixed vertex of $F(\alpha)$. Starting at $P$ trace a closed path $\pi$ around $F(\alpha)$, so moving that $\pi$ does not cross itself, and $\alpha$ is always to the right, and traversing the whole of $F(\alpha)$ terminate the path at $P$. By the number of sides of $\alpha$ we mean the number of arcs in $\pi$, where in this enumeration each arc of $F(\alpha)$ is counted as many times as $\pi$ passes through it.

[10] A 2-cell is called regular if its boundary is a simple closed curve.

Proof: Suppose $H$ subdivides $\mathfrak{S}_K$ quasi-perfectly. By theorem 12 the number of sides of each 2-cell will be $\frac{2\alpha_1}{\alpha_2}$. From the Euler formula we have

$$K = \alpha_0 - \alpha_1 + \alpha_2$$

$$= \alpha_0 - \alpha_1 + \frac{2\alpha_1}{n}$$

$$\therefore \quad n = \frac{2\alpha_1}{\alpha_1 - \alpha_0 + K}. \tag{2}$$

If a graph triangulates a surface, then any other map of the graph on the same surface is a triangulation.[11] This is not true for the more general case of quasi-perfect subdivision. For example, consider the map of the complete 5-point $B$ on a torus. Fig. 2 is a map of this

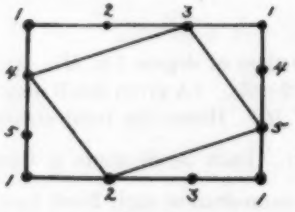


FIG. 2

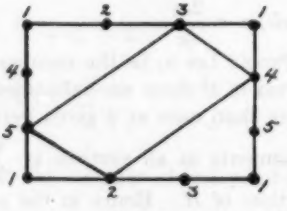


FIG. 3

graph, in which each of the regions is a quadrilateral. But $B$ can also be mapped on a torus as in Fig. 3, in which not all of the regions are quadrilateral, so that this second map is not a quasi-perfect subdivision.

Suppose a graph $H$ is mapped on its maximal surface (i.e. the surface of greatest possible characteristic on which $H$ can be mapped). The surface is then separated into 2-cells.[12] Let $n = \frac{2\alpha_1}{\alpha_2}$. If $H$ subdivides the surface quasi-perfectly then each 2-cell will have $n$ sides (Theorem 12). Suppose $H$ does not subdivide the surface quasi-perfectly. If $\alpha$ is a 2-cell of $(n - \delta)$ sides, we shall say that $\alpha$ has the

[11] T. Theorem 4.

[12] This follows from T. Theorem 2, in the proof of which no assumption was made that the graph considered was free of loops. Theorem 2 of the paper T is valid for any connected graph.

*deficiency* $\delta$. If $\beta$ is a 2-cell of $(n + \epsilon)$ sides, we shall say that $\beta$ has the *excess* $\epsilon$. (If $n$ is not an integer, $\delta$ and $\epsilon$ will, of course, not be integers.)

*Theorem* 14. *Let $H$ be mapped on its maximal surface. Then the sum of all the deficiencies is equal to the sum of all the excesses.*

Proof: Each edge $\xi$ of $H$ is either contained in the set in which the boundaries of two adjacent 2-cells meet, or else the boundary of a single irregular 2-cell meets itself in $\xi$. Hence in an enumeration of the total number of sides of all the 2-cells, each edge is counted twice. That is, the total number of sides is $2\alpha_1$. Let $x$ be the number of 2-cells of less than $n$ sides each, $y$ the number of 2-cells of $n$ sides, and $z$ the number of 2-cells of more than $n$ sides. (If $n$ is not an integer then obviously $y = 0$.) Let $\delta$ be the total deficiency and $\epsilon$ the total excess. Then

$$x + y + z = \alpha_2 = \frac{2\alpha_1}{n}$$

The $x$ 2-cells will have a total of $(nx - \delta)$ sides, while the $z$ 2-cells will have a total of $(nz + \epsilon)$ sides. Therefore the total number of sides is

$$\begin{aligned} 2\alpha_1 &= (nx - \delta) + ny + (nz + \epsilon) \\ &= (nx + ny + nz) - \delta + \epsilon \\ &= 2\alpha_1 - \delta + \epsilon \end{aligned}$$

hence

$$\delta = \epsilon.$$

3. **Quasi-perfect subdivision by complete $N$-points.** The case where $H$ is a complete $N$-point is of special interest. There exists a value $k$ such that for $K \leq k$, the complete $N$-point can be mapped on an $\mathfrak{S}_K$. Let us seek the relationship that must hold between $N$ and $K$ when the complete $N$-point can be mapped on $\mathfrak{S}_K$ so that the surface is subdivided quasi-perfectly.

Suppose a complete $N$-point is mapped in such a manner. Since each vertex is of degree $(N - 1)$, at each vertex there are $(N - 1)$ abutments of 2-cells. The total number of abutments at all vertices is then $N(N - 1)$. Since in this enumeration each 2-cell has been counted $n$ times, once each time it abuts at a vertex, the total number of 2-cells is $\frac{N(N-1)}{n}$. Since the total number of edges is $\frac{N(N-1)}{2}$

we have

$$K = \alpha_0 - \alpha_1 + \alpha_2$$
$$= N - \frac{N(N-1)}{2} + \frac{N(N-1)}{n} \tag{3}$$

whence

$$n = \frac{2N(N-1)}{N^2 - 3N + 2K} \tag{4}$$

Suppose $n$ is held fixed. Solving (3) for $N$ we have

$$N = \frac{(3n-2) \pm \sqrt{(3n-2)^2 - 8nK(n-2)}}{2(n-2)} \tag{5}$$

When $n$ is odd, $(3n-2)$ is odd and the mapping is possible only if $N$ s an integer, that is if

$$(3n-2)^2 - 8nK(n-2)$$

is the square of an odd integer, i.e.

$$(3n-2)^2 + 8nK(2-n) = z^2, \qquad z = 2x+1,$$

that is,

$$z^2 \equiv (3n-2)^2 \pmod{8n(2-n)}$$

or

$$4x(x+1) \equiv 9n^2 - 12n + 3 \pmod{8n(2-n)}.$$

When $n$ is even, $(3n-2)$ is even and the mapping is possible only if

$$(3n-2)^2 - 8nK(n-2)$$

is the square of an even integer, i.e.

$$(3n-2)^2 + 8nK(2-n) = z^2, \qquad z = 2y$$

that is,

$$z^2 \equiv (3n-2)^2 \pmod{8n(2-n)}$$

or

$$y^2 \equiv \left(\frac{3n-2}{2}\right)^2 \pmod{2n(2-n)}.$$

Hence we have proved the two theorems

*Theorem* 15. *If a complete N-point can be mapped on an $\mathfrak{S}_K$ so as to subdivide it quasi-perfectly, then the number of sides of each 2-cell must be n, where n is given by* (4).

*Theorem* 16. *A necessary condition that a complete N-point can be mapped on a surface of characteristic K so as to effect a quasi-perfect subdivision each of the 2-cells having n sides, is that N and K satisfy the relations*

$$\begin{cases} N = \dfrac{(3n-2) \pm z}{2(n-2)} \\ K = \dfrac{z^2 - (3n-2)^2}{8n(2-n)} \end{cases} \tag{6}$$

*where $z = 2x + 1$, and $x$ is any solution of*

$$4x(x+1) \equiv 9n^2 - 12n + 3 \pmod{8n(2-n)}$$

*when n is odd; or $z = 2y$ and $y$ is any solution of*

$$y^2 \equiv \left(\frac{3n-2}{2}\right)^2 \pmod{2n(2-n)}$$

*when n is even.*

*Example*: Consider the set of values $N = 6$, $K = 1$, $n = 4$. These do not satisfy (6). By theorem 16 it follows that a complete 6-point cannot be mapped on a projective plane so as to separate it into 2-cells of four sides each. (This result also follows from the fact that a complete 6-point can be mapped on a projective plane so as to triangulate it, hence by T. Theorem 4 any map of a complete 6-point on a projective plane is a triangulation.)

4. **Perfect Polyhedra.** A graph is called *regular* and of degree $d$ if each of its vertices is of the same degree $d$. Let $G$ be a connected regular graph of degree $\geq 3$ and having no simple loops. We shall call the 2-complex formed by the perfect subdivision of a surface $\mathfrak{S}_K$ by $G$ into 2-cells of $n$ sides each a *perfect polyhedron* of degree $d$, order $n$, and characteristic $K$.

Let $\Pi$ be a perfect polyhedron of characteristic $K$. Then for the number of edges we have $2\alpha_1 = d\alpha_0$. From theorem 12 we have $n = \dfrac{2\alpha_1}{\alpha_2}$. From the Euler formula we then have

$$K = \alpha_0 - \alpha_1 + \alpha_2 = \frac{2\alpha_1}{d} - \alpha_1 + \frac{2\alpha_1}{n}$$

whence

$$\frac{2n - nd + 2d}{nd} = \frac{K}{\alpha_1}. \tag{7}$$

We shall say that two orientable (non-orientable) perfect polyhedra of the same characteristic $K \neq 0$ are of the same *type* if they have the same order and the same degree. From (7) they also have the same number of vertices.

*Theorem* 17. *The number of vertices of a perfect polyhedron of characteristic* $K \neq 0$ *cannot exceed* $14 \mid K \mid$.

Proof: From theorems 24 and 28, which we shall prove in following sections, it shall be evident that the theorem is true for $K = 2$ and $K = 1$. Suppose then that $K < 0$.

From (7) we have, solving for $d$,

$$d = \frac{2n}{n-2}\left(1 - \frac{K}{\alpha_0}\right). \tag{8}$$

From (8), since $d \geq 3$,

$$\frac{2n}{n-2}\left(1 - \frac{K}{\alpha_0}\right) \geq 3$$

whence

$$n \leq \frac{6\alpha_0}{\alpha_0 + 2K}. \tag{9}$$

We shall show that if $\alpha_0 > -14K$, then $d$ cannot be an integer. From (9), if $\alpha_0 > -14K$, then $n \leq \frac{6\alpha_0}{\alpha_0 + 2K} < 7$. Hence we need only consider the values of $d$, as given by (8), for $n = 3, 4, 5, 6$. For example, when $n = 5$, then

$$d = \frac{10}{3}\left(1 - \frac{K}{\alpha_0}\right) = 3\frac{1}{3} - \frac{10K}{3\alpha_0}.$$

Since $\alpha_0 > -14K$, $\frac{-10K}{3\alpha_0} < \frac{2}{3}$ and $d$ is not an integer. A similar reasoning applies for $n = 3, 4, 6$.

It will be seen that this argument fails to hold if $-14K$ is replaced by a smaller integral multiple of $-K$. For if it be merely assumed that $\alpha_0 > -13K$, then from (9), $n < 8$. But when $n = 7$,

$$d = \frac{14}{5}\left(1 - \frac{K}{\alpha_0}\right) = 2\frac{4}{5} - \frac{14K}{5\alpha_0},$$

and if $\alpha_0 = -14K$ then $d = 3$, i.e. is an integer.

*Corollary. If $K \neq 0$, there are a finite number of types of perfect polyhedra of characteristic $K$.*

Proof: From the definition of a perfect polyhedron it is evident that $\alpha_0 > n$. Since $\alpha_0$ has an upper bound, $n$ has an upper bound, and hence $d$ has an upper bound. That is, there are a finite number of values of $(d, n, \alpha_0)$ which satisfy (8), and therefore a finite number of integral values which satisfy (8). Since for any perfect polyhedron $d$, $n$, and $\alpha_0$ must have integral values which satisfy (7), and hence (8), the corollary follows.

Let $\Pi$ be a perfect polyhedron of degree $d$, order $n$ and characteristic $K$. Take a point $p_i$ interior to each 2-cell $\alpha_i$. If two 2-cells $\alpha_j$, $\alpha_k$ have boundaries which meet in an edge $\xi_{jk}$, join $p_j$ to $p_k$ by arc $p_jp_k$ intersecting $\xi_{jk}$. These new arcs, together with the vertices $p_i$ constitute a regular graph $G'$ of degree $n$, having no loops, which subdivides $\mathfrak{S}_K$ perfectly into 2-cells of $d$ sides each. The polyhedron $\Pi'$ thus defined is perfect. We shall call $\Pi'$ the *dual* of $\Pi$. If $\Pi$ is of order $n$ and degree $d$, then the dual of $\Pi$ is of order $d$ and degree $n$. If $d = n$, we shall say the polyhedron $\Pi$ is *self-dual*.

There arise two interesting special cases of formula (7)

*Case I.* $n = d$, self-dual perfect polyhedra.

When $n = d$, (7) becomes

$$\frac{2d - d^2 + 2d}{d^2} = \frac{K}{\alpha_1} = \frac{2K}{d\alpha_0}$$

whence

$$d = 4 - \frac{2K}{\alpha_0} \tag{10}$$

Since $d$ is an integer $\geq 3$, $\alpha_0$ must be a divisor of $2K$, and since $G$ has no loops, $\alpha_0 > d$.

(1) If $K = 2$, the only possible solution of (10) is $\alpha_0 = 4$, $d = 3$. An example is the tetrahedron.

(2) If $K = 1$, the only possible solution of (10) is $\alpha_0 = 2$, $d = 3$. Since in this solution $\alpha_0 < d$, there is no possible polyhedron of characteristic 1.

(3) If $K = 0$, then $d = 4$ and $\alpha_0$ may have any value. Fig. 2 gives an example of an orientable polyhedron of characteristic 0 with $\alpha_0 = 5$. Fig. 4 gives an example of an orientable polyhedron of characteristic 0 with $\alpha_0 = 9$. Similar examples can be constructed with $\alpha_0 = m^2$, where $m$ is any integer $\geq 4$. Fig. 5 gives an example of a non-orientable

polyhedron of characteristic 0 with $\alpha_0 = 9$. Similar examples can be constructed with $\alpha_0 = m^2$, where $m \geq 4$.

(4) For $K = -1$, and $-2$ there are no possible solutions of (10).

(5) For $K \leq -3$, $d \geq 4 + \frac{6}{\alpha_0}$. That is, $d \geq 5$.

We summarize these results in the theorems;

*Theorem* 18. *The only self-dual perfect polyhedron of characteristic* 2 *is the tetrahedron. No self dual perfect polyhedra of characteristic* $-1$, $-1$, 2 *exist.*

*Theorem* 19. *If a self-dual perfect polyhedron of characteristic* $\leq -3$ *exists, its degree is* $\geq 5$.

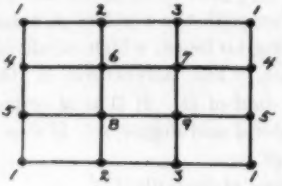

FIG. 4

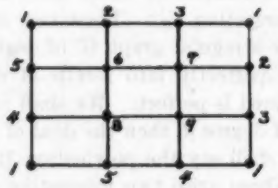

FIG. 5

A table of possible solutions of (10) up to $K = -12$ follows,

| $K$ | $\alpha_0$ | $d = n$ | $K$ | $\alpha_0$ | $d = n$ |
|---|---|---|---|---|---|
| 2 | 4 | 3 | $-8$ | 8 | 6 |
| | | | | 16 | 5 |
| 0 | | 4 | | | |
| | | | $-9$ | 9 | 6 |
| $-3$ | 6 | 5 | | 18 | 5 |
| $-4$ | 8 | 5 | $-10$ | 10 | 6 |
| | | | | 20 | 5 |
| $-5$ | 10 | 5 | | | |
| | | | $-11$ | 11 | 6 |
| $-6$ | 12 | 5 | | 22 | 5 |
| $-7$ | 7 | 6 | $-12$ | 8 | 7 |
| | 14 | 5 | | 12 | 6 |
| | | | | 24 | 5 |

*Case II.* $n = 3$, triangulation of surfaces by regular graphs.

(The dual of this case arises when $d = 3$, the perfect subdivision of surfaces by regular graphs of degree three and having no loops. Such subdivisions are also called regular maps.)

When $n = 3$, (7) becomes

$$\frac{6 - 3d + 2d}{3d} = \frac{K}{\alpha_1} = \frac{2K}{d\alpha_0}$$

whence

$$d = 6 - \frac{6K}{\alpha_0} \tag{11}$$

Since $d$ is an integer $\geq 3$, $\alpha_0$ must be a divisor of $6K$, and since $G$ has no loops, $\alpha_0 > d$. A table of possible solutions of (11) up to $K = -4$ follows,

| $K$ | $\alpha_0$ | $d$ | $K$ | $\alpha_0$ | $d$ |
|---|---|---|---|---|---|
| 2 | 4 | 3 | −1 | No possible solutions | |
| | 6 | 4 | | | |
| | 12 | 5 | −2 | 12 | 7 |
| 1 | 6 | 5 | −3 | 9 | 8 |
| | | | | 18 | 7 |
| 0 | — | 6 | | | |
| | | | −4 | 12 | 8 |
| | | | | 24 | 7 |

*Examples*

(1) $K = 2$, $\alpha_0 = 4$, $d = 3$, the tetrahedron is an example.

$\alpha_0 = 6$, $d = 4$, the octahedron is an example, (the dual is a cube).

$\alpha_0 = 12$, $d = 5$, the icosahedron is an example, (the dual is a dodeckahedron.

(2) $K = 1$. The only possible solution is $\alpha_0 = 6$, $d = 5$. An example is given in fig. 6.

(3) $K = 0$. Here $d = 6$, and $\alpha_0$ can have any value. We give examples of regular graphs with seven, and with eight vertices, which triangulate the torus, (figs. 7 and 8 respectively). In fig. 1 we have an example of a regular graph with sixteen vertices which triangulates

the torus. We can construct examples similar to the latter with $\alpha_0 = m^2$, where $m$ is any positive integer $\geq 3$.

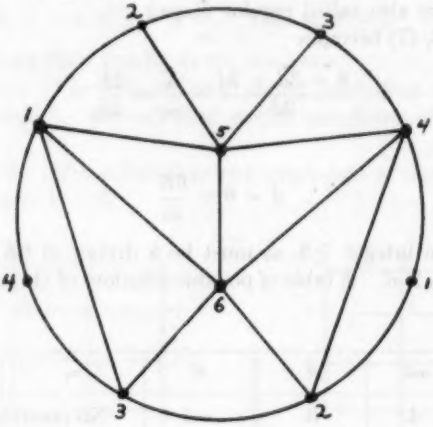


FIG. 6

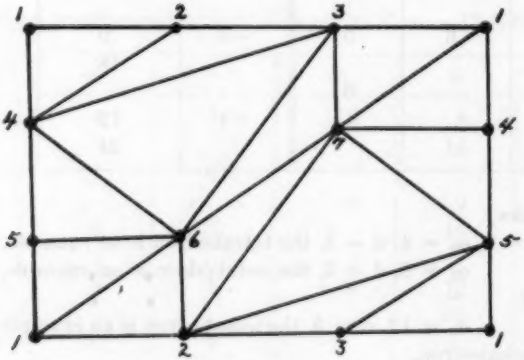


FIG. 7

In fig. 9 we give an example of a regular graph which triangulates a non-orientable surface of characteristic 0. Similar examples can be constructed for $\alpha_0 = m^2$, where $m \geq 3$.

(4) For $K = -1$, there are no possible solutions of (11).

(5) $K = -2$. $\alpha_0 = 12, d = 7$. In fig. 10 we give an example for the orientable surface of characteristic $-2$.

*Theorem* 20. *No regular graph exists which triangulates a surface of characteristic* $-1$.

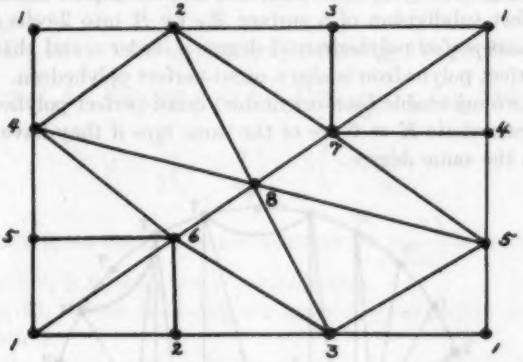


FIG. 8

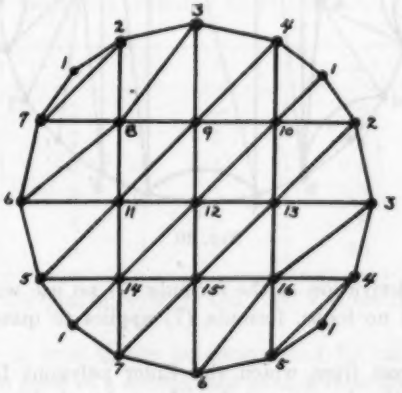


FIG. 9

*Theorem* 20′ (*dual of theorem* 20). *No regular graph of degree three exists which subdivides a surface of characteristic* $-1$ *perfectly.*

*Theorem* 21. *If a regular graph triangulates a surface of characteristic* $K \leq -2$ *its degree is* $\geq 7$.

Proof: For $K \leq -2, d \geq 6 + \frac{12}{\alpha_0}$. That is $d \geq 7$.

5. **Quasi-perfect polyhedra.** Let $H$ be any finite connected regular graph of degree $d \geq 3$. We shall call the 2-complex formed by the quasi-perfect subdivision of a surface $\mathfrak{S}_K$ by $H$ into 2-cells of $n$ sides each, a *quasi-perfect* polyhedron of degree $d$, order $n$ and characteristic $K$. A perfect polyhedron is also a quasi-perfect polyhedron. We shall say that two orientable (non-orientable) quasi-perfect polyhedra of the same characteristic $K \neq 0$ are of the same *type* if they have the same order and the same degree.

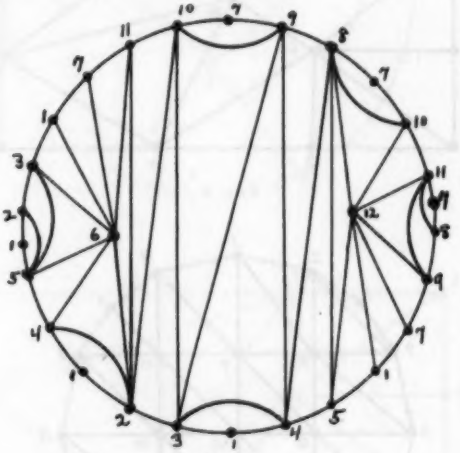

FIG. 10

Since in the derivation of the formula (7) no use was made of the fact that $G$ had no loops, formula (7) applies to quasi-perfect polyhedra also.

The 2-complexes from which the Euler polygons for the various surfaces are obtained are examples of quasi-perfect polyhedra which are not perfect. Additional examples are given in fig. 13.

6. **Spherical polyhedra.** When $K = 2$, (7) becomes

$$\frac{2n - nd + 2d}{nd} = \frac{2}{\alpha_1} = \frac{4}{d\alpha_0},$$

solving for $d$,

$$d = \frac{n}{n-2}\left(2 - \frac{4}{\alpha_0}\right). \tag{12}$$

*Theorem* 22. *The order of a quasi-perfect polyhedron of characteristic* 2 *cannot exceed* 5.

Proof: Let $M$ be the regular graph defining the polyhedron. By definition $d \geq 3$. Suppose $n \geq 6$. Then $\frac{n}{n-2} \leq \frac{3}{2}$ and from (12)

$$d \leq \frac{3}{2}\left(2 - \frac{4}{\alpha_0}\right) = 3\left(1 - \frac{2}{\alpha_0}\right).$$

Now for any finite value of $\alpha_0$ the quantity $\left(1 - \frac{2}{\alpha_0}\right)$ is less than unity and hence $d < 3$, which gives a contradiction.

*Theorem* 23. *There exist only six types of quasi-perfect polyhedra of characteristic* 2.

Proof: By theorem 22, $n \leq 5$. Giving $n$ successively the values 2, 3, 4, 5 we find the following table of solutions of (12), (where the values of $\alpha_2$ are computed from the Euler formula).

| Type | $n$ | $d$ | $\alpha_0$ | $\alpha_2$ |
|---|---|---|---|---|
| Q | 2 | — | 2 | — |
| I | 3 | 3 | 4 | 4 |
| II | 3 | 4 | 6 | 8 |
| III | 3 | 5 | 12 | 20 |
| IV | 4 | 3 | 8 | 6 |
| V | 5 | 3 | 20 | 12 |

(Type IV is the dual of type II, and type V is the dual of type III.)

An example of type Q can be constructed by taking for the two vertices the two poles of a sphere, and joining these by a finite number of arcs. The faces will be lunes. The graphs defining this type of polyhedra have loops, so that the polyhedra are not perfect. Since examples of types I–V are given by I) tetrahedron, II) octahedron, III) icosahedron, IV) cube, and V) dodeckahedron, we have the

*Theorem* 24. *There exist only five types of perfect polyhedra of characteristic* 2.

Two 2-complexes are said to be *isomorphic* if we can set up a 1–1 correspondence between the 0-, 1-, and 2-cells of the two complexes such that incidence relations are preserved.

*Theorem* 25. *Any two spherical perfect polyhedra of the same type are isomorphic.*

Proof: We shall give the proof for type III. The proof for types I and II is similar, and somewhat simpler. Since type V is the dual of type III, and type IV the dual of type II, the theorem follows for types IV and V, under duality.[12]

Let $\pi$ be any perfect polyhedron of type III. We shall denote the 2-star of a vertex $x$ by the symbol $\sigma_x$, and the boundary of the 2-star by $F(\sigma_x)$. Let $x$ be any vertex of $\pi$. Since the 2-cells of $\pi$ are all triangles, and each vertex is of degree 5, the 2-star of any vertex of $\pi$ is bounded by a simple closed curve of five arcs and five vertices, (theorem 2). Hence we can assign the letters $y_1$, $y_2$, $y_3$, $y_4$, $y_5$ in cyclic order to the vertices of $F(\sigma_x)$ starting with any vertex. For each triangle $xy_iy_{i+1}$ of $\sigma_x$ there is a triangle $z_{i,i+1}y_iy_{i+1}$ of $\pi$ whose boundary has the arc $\overset{\frown}{y_iy_{i+1}}$ in common with $F(\sigma_x)$, (where $i = 1, \cdots, 5$, and the additions in the subscripts are understood to be modulo 5). By the *flower* $f_x$ of $x$ we shall mean the 2-complex consisting of $\sigma_x$, $F(\sigma_x)$, and the triangles $z_{i,i+1}y_iy_{i+1}$ together with their boundaries. $f_x$ contains ten triangles. We shall first prove that the vertices on the boundary of $f_x$ are all distinct.

$F(\sigma_{y_i})$ contains the chain $z_{i-1,i}y_{i-1}xy_{i+1}z_{i,i+1}$. If $z_{i-1,i}$ were identical with $z_{i,i+1}$, $F(\sigma_{y_i})$ would be the circuit $z_{i-1,i}y_{i-1}xy_{i+1}z_{i-1,i}$, which is impossible, since this circuit contains only four arcs. Hence $z_{i-1,i} \neq z_{i,i+1}$ and $F(\sigma_{y_i}) = z_{i-1,i}y_{i-1}xy_{i+1}z_{i,i+1}z_{i-1,i}$. Thus the symbols $\overset{\frown}{z_{i-1,i}z_{i,i+1}}$ represent arcs of $\pi$. $F(\sigma_{z_{i,i+1}})$ contains the chain $z_{i-1,i}y_iy_{i+1}z_{i+1,i+2}$. If $z_{i-1,i}$ were identical with $z_{i+1,i+2}$, $F(\sigma_{z_{i,i+1}})$ would be the circuit $z_{i-1,i}y_iy_{i+1}z_{i-1,i}$ which is impossible since this circuit contains only three arcs. Hence $z_{i-1,i} \neq z_{i+1,i+2}$. It follows that the $z$'s are all distinct. Since $z_{i-1,i}$ is not contained in $F(\sigma_x)$, $z_{i-1,i}$ is distinct from each of the $y$'s.

[12] If $\Pi'$ is the dual of $\Pi$, then there is a 1-1 correspondence between the 0-, 1-, 2-cells of $\Pi$ and the 2-, 1-, 0-cells of $\Pi'$. If $\Pi''$ is isomorphic with $\Pi'$ then there is a 1-1 correspondence between the 2-, 1-, 0-cells of $\Pi''$ and the 2-, 1-, 0-cells of $\Pi'$. If $\Pi'''$ is the dual of $\Pi''$ then there is a 1-1 correspondence between the 2-, 1-, 0-cells of $\Pi''$ and the 0-, 1-, 2-cells of $\Pi'''$. Thus there is a 1-1 correspondence between the 0-, 1-, 2-cells of $\Pi'''$ and the 0-, 1-, 2-cells of $\Pi$. Hence if $\Pi'$ is isomorphic with $\Pi''$, then the dual of $\Pi'$ is isomorphic with the dual of $\Pi''$.

By the *constellation* $k_x$ of $x$ we shall mean the 2-complex consisting of the flower $f_x$, and the triangles $z_{i-1,i}yz_{i,i+1}$ together with their boundaries. $k_x$ contains fifteen triangles. The boundary of $k_x$ is the circuit $z_{13}z_{23}z_{34}z_{45}z_{51}z_{12}$.

Let II be any other perfect polyhedron of type III, $X$ any vertex of II. Let $k_X$ be the constellation of $X$, with vertices $X$, $Y_i$, $Z_{i,i+1}$, ($i = 1, \cdots, 5$, and the addition in the subscripts being understood to be modulo 5), constructed as is $k_x$ by first adding to $\sigma_X$ the closed triangles $Z_{i,i+1}Y_iY_{i+1}$ and then adding the closed triangles $Z_{i-1,i}Y_iZ_{i,i+1}$. As above, we can show that the vertices of $F(k_X)$ are all distinct. We can set up an isomorphism between $k_x$ and $k_X$, letting each element of $k_x$ correspond to the element of $k_X$ obtained by replacing small letters by capital letters in the symbols for them.

For each arc $z_{i-1,i}z_{i,i+1}$ of $F(k_x)$ there is a triangle $\overset{\frown}{z_{i-1,i}z_{i,i+1}}w_i$ whose boundary has the arc $\overset{\frown}{z_{i-1,i}z_{i,i+1}}$ in common with $F(k_x)$. Now $k_x$ contains eleven vertices, and since the total number of vertices of $\pi$ is twelve, the vertices $w_i$ are identical. Denote this vertex by $w$. Then $F(\sigma_w)$ is identical with $F(k_x)$, namely $z_{13}z_{23}z_{34}z_{45}z_{51}z_{12}$. Similarly II contains a vertex $W$ such that $F(\sigma_W) = F(k_X) = Z_{12}Z_{23}Z_{34}Z_{45}Z_{51}Z_{12}$. Hence we can let $\sigma_w$ correspond to $\sigma_W$, the arcs $wz_{i,i+1}$ corresponding to $WZ_{i,i+1}$, and the triangles $wz_{i-1,i}z_{i,i+1}$ to $WZ_{i-1,i}Z_{i,i+1}$.

Thus we have established an isomorphism between $\pi$ and II.

From theorem 25 we see that every perfect polyhedron of type I is isomorphic with the regular tetrahedron, etc. Hence if we restrict our attention to the regular polyhedra, theorem 24 becomes the classic theorem of Euclidean Solid Geometry; "There exist only five kinds of regular polyhedra." Since theorem 24 was arrived at by purely topological considerations, we have a non-metrical proof of the Euclidean theorem.

7. **Projective-planar polyhedra.** When $K = 1$, (7) becomes

$$\frac{2n - nd + 2d}{nd} = \frac{1}{\alpha_1} = \frac{2}{d\alpha_0}$$

solving for $d$,

$$d = \frac{n}{n-2}\left(2 - \frac{2}{\alpha_0}\right) \tag{13}$$

*Theorem* 26. *The order of a quasi-perfect polyhedron of characteristic* 1 *cannot exceed* 5.

The proof is analogous to that of theorem 22.

*Theorem* 27. *There exist only five types of quasi-perfect polyhedra of characteristic* 1.

Proof: By theorem 24, $n \leq 5$. Putting $n = 2, 3, 4, 5$ successively in (13), we find the following table of solutions of (13) (rejecting those for which $d < 3$),

| Type | $n$ | $d$ | $\alpha_2$ |
|---|---|---|---|
| $Q_1$ | 2 | — | 1 |
| — | 3 | 3 | 2 |
| $Q_2$ | 3 | 4 | 3 |
| I | 3 | 5 | 6 |
| $Q_3$ | 4 | 3 | 4 |
| II | 5 | 3 | 10 |

Type II is the dual of type I.

There are six solutions of (13) in this table. But consider the solution $n = 3$, $d = 3$, $\alpha_0 = 2$. If such a polygon existed, the defining graph would be one of the two following

(a) Three arcs each joining the same pair of vertices $a$, $b$; or

(b) Two circuits $\lambda_1$, $\lambda_2$ and an arc $cd$, where $c \in \lambda_1$, $d \in \lambda_2$. Neither of these two graphs can be mapped on the projective plane so as to subdivide it quasi-perfectly, (which is easily verified empirically). As examples of types $Q_1$, $Q_2$, $Q_3$, I, II are given below, the theorem is proved.

An example of type $Q_1$ can be constructed by taking for the defining graph a finite number ($\geq 2$) of concurrent projective lines. The faces will be 2-sided 2-cells. Since the graph has loops, the example is a quasi-perfect, but not a perfect polyhedron.

An example of type $Q_2$ is the 2-complex formed by three non-concurrent projective lines on the projective plane. Since the graph defining the polyhedron has loops, this polyhedron is quasi-perfect, but not perfect.

An example of type I is given in fig. 6. By constructing the dual of this polyhedron we obtain an example of type II.

Now consider type $Q_3$. We shall show that there exists no perfect polyhedra of this type. Let II be any polyhedron of type $Q_3$, $x$ any vertex of II. We shall suppose II is perfect, and force a contradiction. Since $x$ is of degree three, $F(\sigma_x)$ will contain three vertices adjacent to $x$,

to which we can assign the letters $y_1$, $y_2$, $y_3$ in cyclic order. The $y_i$ are all distinct, since otherwise the defining graph of $\Pi$ would contain a loop, contrary to the definition of a perfect polyhedron. Since $x$ is of degree three, three quadrilaterals abut at $x$, namely the quadrilaterals $xy_iz_{i,i+1}y_{i+1}$, (where $i = 1, 2, 3$, and the additions in the subscripts are understood to be modulo 3). Since $\Pi$ contains four vertices, each $z$ must be identical with some $y$. Now $z_{i,i+1}$ is adjacent to $y_i$ and $y_{i+1}$, hence is distinct from them. Hence $z_{i,i+1} = y_{i+2}$, and $\Pi$ consists of the quadrilaterals $xy_iy_{i+2}y_{i+1}$ together with their boundaries. The boundary of the quadrilateral $xy_iy_{i+2}y_{i+1}$ has the arcs $xy_i$ and $y_{i+2}y_{i+1}$ in common with the boundary of the quadrilateral $xy_{i+2}y_{i+1}y_i$. But by definition of a perfect polyhedron, the boundaries of any two 2-cells can meet in at most a connected set. Hence $\Pi$ is not a perfect polyhedron.

Since for a perfect polyhedron we must have $\alpha_0 > d$, (for the graph defining the polyhedron contains no loops), and for each of the types $Q_1$ and $Q_2$ $\alpha_0 < d$, there exist no perfect polyhedra of these types. Hence

*Theorem* 28. *There exist only two types of perfect polyhedra of characteristic* 1.

*Theorem* 29. *Any two perfect polyhedra of characteristic* 1 *which are of the same type are isomorphic.*

The proof follows along the same general lines as that of theorem 25.

8. **Polyhedra of characteristic** 0. When $K = 0$, (7) becomes

$$\frac{2n - nd + 2d}{nd} = 0$$

$$2n - nd + 2d = 0. \tag{14}$$

The only possible solutions of (14) are

$$\left.\begin{matrix} n = 3, \\ d = 6, \end{matrix}\right\} \quad \left.\begin{matrix} 4, \\ 4, \end{matrix}\right\} \quad \left.\begin{matrix} 6. \\ 3. \end{matrix}\right\}$$

Hence

*Theorem* 30. *If* $\Pi$ *is a quasi-perfect polyhedron of characteristic zero, the faces of* $\Pi$ *are either triangles, quadrilaterals, or hexagons.*

*Examples*: (A) Perfect polyhedra.

(1) $n = 3$, $d = 6$. Fig. 7 gives an example of an orientable polyhedron and fig. 9 gives an example of a non-orientable polyhedron.

(2) $n = 4$, $d = 4$. Fig. 2 gives an example of an orientable polyhedron and fig. 5 an example of a non-orientable polyhedron.

(3) $n = 6$, $d = 3$. Fig. 11 gives an example of an orientable polyhedron and fig. 12 an example of a non-orientable polyhedron.

(B) Quasi-perfect, non-perfect polyhedra.

Examples are given in fig. 13.

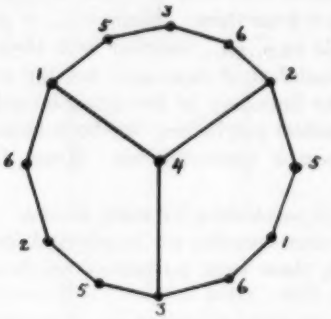

FIG. 11

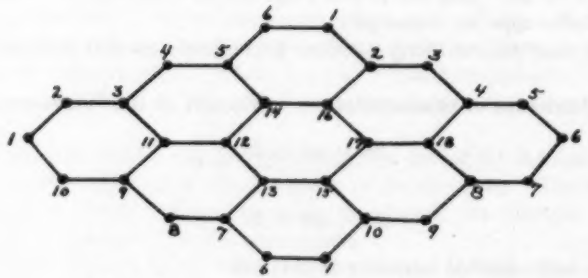

FIG. 12

9. **Infinite perfect subdivision of a plane.** In this and the following section we consider an infinite graph lying on an open surface and having the properties; every arc is closed, every vertex is of finite degree, and no infinite subset of arcs of the graph form a set which is compact relative to the surface. If every vertex has the same degree we shall say the graph is regular. By an *infinite perfect subdivision* of the surface we shall mean the separation of the surface by an infinite graph into

regular 2-cells, each of which is compact, and each of which has the same number $n$ of sides, and such that the boundaries of any two of these 2-cells meet in at most a connected set.

*Theorem* 31. *If $n$ and $d$ has any pair of values distinct from*

$$\begin{array}{lccccc} n = 3, & 3, & 3, & 4, & 5. \\ d = 3, & 4, & 5, & 3, & 3. \end{array}$$

*then there exists an infinite perfect subdivision of the plane of degree $d$ and order $n$.*

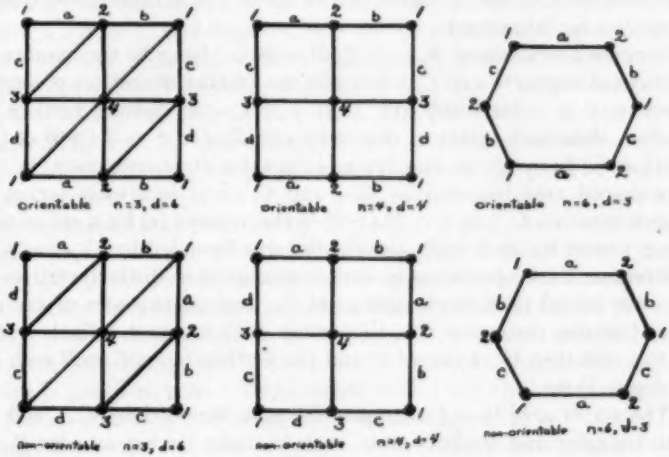


FIG. 13

Proof: There arise three possibilities; I, $n > 3$, $d > 3$; II, $n = 3$, $d > 5$; and III, $n > 5$, $d = 3$.

*Case I*, $n > 3$, $d > 3$.

Let $x$ be any point of the plane. Let $[C_r]$ be a system of concentric circles with center at $x$ and radii $r$, $(r = 0, 1, \cdots)$, where $C_0$ is the point $x$. Let $y_1, \cdots, y_d$ be $d$ distinct points of $C_1$. Join $x$ to each $y_i$ $(i = 1, \cdots, d)$. Introduce $n - 3$ $(>0)$ new vertices $z_{ij}$, $(j = 1, \cdots, n - 3)$ on each arc $\widehat{y_i y_{i+1}}$ of $C_1$, (where the additions in the subscripts are understood to be modulo $d$). The 2-star of $x$ now consists of $d$ $n$-gons. On $C_1$ each $y$ is of degree three, and each $z$ is of degree two.

Let $S_1$ denote the subdivision of the interior of $C_1$ by the graph $x + \Sigma\widehat{xy_i} + C_1$ and $S_0$ denote the point $x$. We shall show that there exists a subdivision $S_r$, $(r \geq 2)$ of the interior $C_r$, by a graph $G_r$ including $C_r$, such that (1) $S_r$ is obtained from $S_{r-1}$ by introducing vertices on $C_r$ and joining each vertex of $C_{r-1}$ to certain vertices of $C_r$ by arcs lying, except for their endpoints, in the ring-shaped area bounded by $C_{r-1}$ and $C_r$, in such a manner that distinct vertices of $C_{r-1}$ are joined to distinct vertices of $C_r$, and no two of the arcs joining vertices of $C_{r-1}$ to vertices of $C_r$ intersect; (2) each 2-cell of $S_r$ has $n$ sides; (3) each vertex of $G_r$ which is not on $C_r$ is of degree $d$; and (4) the vertices of $C_r$ are of degree two or three. $S_1$ satisfies these conditions in a trivial manner.

Suppose a subdivision $S_{r-1}$, $(r \geq 2)$, exists. Let $p$ be the number of vertices of degree two on $C_{r-1}$ and $q$ be the number of vertices of degree three on $C_{r-1}$. Introduce $p(d-2) + q(d-3)$ distinct vertices $[v]$ on $C_r$. Join each vertex of degree two on $C_{r-1}$ to $d-2$ $(>0)$ of the vertices $[v]$ by a set of arcs lying, except for their endpoints, in the ring-shaped area bounded by $C_{r-1}$ and $C_r$, and join each vertex of degree three on $C_{r-1}$ to $d-3$ $(>0)$ of the vertices $[v]$ by a set of arcs lying, except for their endpoints, in the area bounded by $C_{r-1}$ and $C_r$; performing these operations in such a manner that distinct vertices of $C_{r-1}$ are joined to distinct vertices of $C_r$, and no two arcs of the set $[\xi_{r-1,r}]$ joining vertices of $C_{r-1}$ to vertices of $C_r$ intersect. Each vertex of $C_{r-1}$ will then be of degree $d$, and the vertices $[v]$ on $C_r$ will each be of degree three.

The set of arcs $[\xi_{r-1,r}]$ separates the area bounded by $C_{r-1}$ and $C_r$ into triangles and quadrilaterals, each triangles having an edge $\beta_t$ on $C_r$, and each quadrilateral having an edge $\beta_q$ on $C_r$. Introduce $n-3$ $(>0)$ new vertices $[w_t]$ on each arc $\beta_t$ and $n-4$ $(>0)$ new vertices $[w_q]$ on each arc $\beta_q$. When this is done, the area bounded by $C_{r-1}$ and $C_r$ will be separated into $n$-gons, and each of the vertices $[w]$ on $C_r$ will be of degree two. We thus have constructed an $S_r$.

It is clear that the subdivision

$$S_0 + (S_1 - S_0) + (S_2 - S_1) + \cdots + (S_r - S_{r-1}) + \cdots$$

has the desired properties.

*Case II.* $n = 3$, $d > 5$.

Let $x$ and the set of circles $[C_r]$ be defined as in case I. Let $y_1, \cdots, y_d$ be $d$ distinct points of $C_1$. Join $x$ to each $y_i$. The 2-star of $x$ consists of $d$ triangles. On $C_1$ each vertex is of degree three.

Let $S_1$ denote the subdivision of the interior of $C_1$ by the graph $x + \Sigma\widehat{xy}_i + C_1$, and $S_0$ denote the point $x$. We shall show that there exists a subdivision $S_r$ ($r \geq 2$), of the interior of $C_r$, by a graph $G_r$ including $C_r$, such that (1) $S_r$ is obtained from $S_{r-1}$ by introducing vertices on $C_r$ and joining each vertex of $C_{r-1}$ to certain vertices of $C_r$ by arcs lying, except for their endpoints, in the ring-shaped area bounded by $C_{r-1}$ and $C_r$, in such a manner that each vertex of $C_{r-1}$ is joined to two or more vertices of $C_r$, and for each pair of adjacent vertices of $C_{r-1}$ there exists a single vertex on $C_r$ which is joined to both; (2) no two of the arcs joining vertices of $C_{r-1}$ to vertices of $C_r$ intersect; (3) each 2-cell of $S_r$ has three sides; (4) each vertex of $G_r$ which is not on $C_r$ is of degree $d$; and (5) the vertices of $C_r$ are of degree three or four. $S_1$ satisfies these conditions in a trivial manner.

Suppose a subdivision $S_{r-1}$ exists ($r \geq 2$). Let $p$ be the number of vertices of degree three on $C_{r-1}$, and $q$ be the number of vertices of degree four on $C_{r-1}$. Introduce $p(d-3) + q(d-4)$ distinct vertices $[z]$ on $C_r$. Join each vertex of degree three on $C_{r-1}$ to $d-3$ ($>0$) of the vertices $[z]$ by a set of arcs lying, except for their endpoints, in the ring-shaped area bounded by $C_{r-1}$ and $C_r$, and join each vertex of degree four on $C_{r-1}$ to $d-4$ ($>0$) of the vertices $[z]$ by a set of arcs lying, except for their endpoints, in the area bounded by $C_{r-1}$ and $C_r$; performing these operations in such a manner that no two of the arcs $[\xi_{r-1,r}]$ joining vertices of $C_{r-1}$ to vertices of $C_r$ intersect, and such that for each pair of adjacent vertices of $C_{r-1}$ there is a single vertex of $C_r$ which is joined to both. Each vertex of $C_{r-1}$ will then be of degree $d$.

The set of arcs $[\xi_{r-1,r}]$ triangulates the area bounded by $C_{r-1}$ and $C_r$. Assign the letters $w_1, w_2, \cdots, w_j, \cdots$ in cyclic order to the vertices of $C_{r-1}$. $w_{j-1}$ and $w_j$ are each joined to the same vertex $z_{j-1,j}$ of $C_r$, and $w_j$ and $w_{j+1}$ are each joined to the same vertex $z_{j,j+1}$ of $C_r$. Since $d > 5$, and $w_j$ is of degree three or four, $w_j$ is joined to at least two vertices of $C_r$, hence for each $j$, $z_{j-1,j} \neq z_{j,j+1}$. Therefore each $z_{j,j+1}$ is joined to the vertices $w_j$ and $w_{j+1}$ of $C_{r-1}$ and no others, hence is of degree four. Thus each vertex of $C_r$ which is joined to an adjacent pair of vertices of $C_{r-1}$ is of degree four. The other vertices of $C_r$ are of degree three. We thus have constructed an $S_r$.

It is clear that the subdivision

$$S_0 + (S_1 - S_0) + (S_2 - S_1) + \cdots + (S_r - S_{r-1}) + \cdots$$

has the desired properties.

*Case III.* $n > 5, d = 3$.

This case is the dual of case II, hence by taking the dual of the subdivision of case II we obtain a subdivision having the desired properties.

*Theorem* 32. *If $n$ and $d$ have any of the five pairs of values listed in the hypothesis of theorem* 31, *then there exists no infinite perfect subdivision of the plane of degree $d$ and order $n$.*

Proof: We shall prove the theorem for the case $n = 3, d = 5$. The proof for $n = 3$, $d = 3$, and for $n = 3$, $d = 4$ is similar. The proof for $n = 4$, $d = 3$, and for $n = 5$, $d = 3$ then follows by duality.

Suppose there exists an infinite graph $G$ which effects an infinite perfect subdivision of the plane of degree 5 and order 3. Let $x$ be any vertex of $G$. Consider the constellation $k_x$ of $x$. As in theorem 25 we can show that the boundary $F(k_x)$ of $k_x$ is a circuit of five arcs and five vertices. Let $H$ be the graph consisting of the 0- and 1-cells of $k_x$. On $H$ each vertex of $F(k_x)$ is of degree four. Since each of the 2-cells of the subdivision of the plane is a triangle, for each pair of adjacent vertices of $F(k_x)$ there exists a single vertex of $G - H$ which is joined to both. Since on $H$ each vertex of $F(k_x)$ is of degree four, it is joined to one vertex of $G - H$. Hence all the vertices of $F(k_x)$ must be joined to a single vertex $w$ of $G - H$. Let $M$ be the graph consisting of $H$, $w$, and the set of arcs joining vertices of $F(k_x)$ to $w$. Since there are five vertices on $F(k_x)$, $w$ is of degree five on $M$. Since $G$ is infinite, $G - M \neq 0$. Hence $w$ is joined to some vertex of $G - M$. But then $w$ is of degree greater than five, which is impossible. Hence an infinite perfect subdivision having the desired properties does not exist.

10. **A perfect subdivision of covering surfaces.** Let $\Sigma_p$ be an orientable surface of genus $p$ $(>0)$, $U_p$ the universal covering surface of $\Sigma_p$, and $\{t\}$ the group of decktransformations of $U_p$. If $u$, $v$, are two points of $U_p$ which cover a point $z$ of $\Sigma_p$, then there is a transformation $t$ of $\{t\}$ such that $t(u) = v$, and any non-identical transformation of $\{t\}$ carries a point $u$ of $U_p$ into a point of $U_p$ which covers the same point of $\Sigma_p$ that $u$ does, i.e. leaves no points of $U_p$ fixed. Two points $u_1$, $u_2$ of $U_p$ shall be called *congruent* if there exists a decktransformation $t$ such that $t(u_1) = u_2$, i.e. if $u_1$ and $u_2$ cover the same point of $\Sigma_p$. A set of points $\mathfrak{D}$ of $U_p$ shall be called a *fundamental domain* of $\{t\}$, if for every point $u$ of $U_p$ there is one and only one point of $\mathfrak{D}$ congruent to $u$, and if no two points of $\mathfrak{D}$ are congruent to each other, (i.e. each point of $\Sigma_p$ is covered by one and only one point of $\mathfrak{D}$, and no two points of $\mathfrak{D}$ cover the same point of $\Sigma_p$). The set of points $\mathfrak{D}$ covers

$\Sigma_p$ exactly once. Every transformation of $\{t\}$ transforms a fundamental domain into another fundamental domain. By a *proper* fundamental domain we shall mean a fundamental domain $\mathfrak{D}$ which consists of a 2-cell $\delta$, bounded by a graph $G$ having no free arcs or chains, together with a suitably chosen subset of $G$, (where by a free arc or chain of a graph we mean an arc or chain one of whose ends is a vertex of degree one of the graph).

Let $\mathfrak{D}$ be a proper fundamental domain of the group $\{t\}$ of deck-transformations of $U_p$, and let $\overline{\mathfrak{D}}$ denote the closure of $\mathfrak{D}$. The set of points of $G$ plus all points congruent to points of $G$ forms a network which is an infinite graph $D$ having no free arcs or chains. $U_p - D$ consists of an infinite number of 2-cells, $\delta_j$, each of which together with a suitably chosen subset of its boundary is a proper fundamental domain of $U_p$. We shall show that $D$ subdivides $U_p$ perfectly.

$G$ is a simple closed curve, for suppose this were not so. Then $\overline{\mathfrak{D}}$ is multiply connected. Let $\lambda$ be the outer boundary[14] of $\mathfrak{D}$, and $\bar{\Delta}$ the simply connected set consisting of $\lambda$ plus its interior $\Delta$. It is easily seen that $\bar{\Delta} - \overline{\mathfrak{D}}$ is a non-vacuous open set, the boundary points of which are ~~continued~~ contained in the set of boundary points of $\overline{\mathfrak{D}}$. Let $x$ be any point of $\bar{\Delta} - \overline{\mathfrak{D}}$. There exists a point $x_0$ of $\mathfrak{D}$, and a decktransformation $t$ such that $t(x_0) = x$. Thus at least one point of $t(\overline{\mathfrak{D}})$ lies in $\bar{\Delta} - \overline{\mathfrak{D}}$. $t(\overline{\mathfrak{D}})$ is contained in $\overline{\bar{\Delta} - \overline{\mathfrak{D}}}$, for if not, then since $t(\overline{\mathfrak{D}})$ and $\overline{\bar{\Delta} - \overline{\mathfrak{D}}}$ are the closures of open sets, and in particular $t(\overline{\mathfrak{D}})$ is the closure of a connected open set, it is clear that some inner point of $t(\overline{\mathfrak{D}})$ coincides with a boundary point of $\bar{\Delta} - \overline{\mathfrak{D}}$, hence of $\overline{\mathfrak{D}}$, which is impossible. Therefore $t(\lambda) \subset \bar{\Delta} - \overline{\mathfrak{D}} \subset \bar{\Delta}$ so that $t(\lambda)$ is interior to $\lambda$, and $t(\bar{\Delta})$ is interior to $\bar{\Delta}$. But then by a theorem of Brouwer, $t$ must leave at least one point of $\bar{\Delta}$ fixed, which is impossible. Hence $G$ is a simple closed curve, and $\delta$ is a regular 2-cell.

Let $n$ be the number of arcs of the network $D$ which make up $G$. We shall say that $\mathfrak{D}$ has $n$ sides.

Two proper fundamental domains $\mathfrak{D}_1$, $\mathfrak{D}_2$ shall be called *adjacent* if their boundaries $G_1$, $G_2$ meet. If $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are adjacent then $G_1 \cap G_2$ is connected. For if this were not so, then $\mathfrak{D}_1 + \mathfrak{D}_2$ would be multiply connected with an outer boundary $\mu$. But then by reasoning similar

[14] By the outer boundary of $\mathfrak{D}$ we mean the set of boundary points of $\mathfrak{D}$ which is accessible from infinity.

to the preceding paragraph we can show that this leads to a contradiction.

Each vertex of $D$ is of degree $\geq 3$. Since each $G_i$ is the image of $G$ under a transformation of $\{t\}$, each 2-cell $\delta_j$ has the same number of sides as $\delta$ and is a regular 2-cell.

Hence we have shown that $D$ subdivides $U_p$ perfectly.

Let $H$ be a connected graph containing no free arcs or chains, which is mapped on a $\Sigma_p$ in such a manner that $\Sigma_p - H$ is a single 2-cell $\beta$. Then the covering surface $U_p$ contains a network $H^*$ covering $H$, which is an infinite graph containing no free arcs or chains, and $U_p - H$ is an infinite set of 2-cells $\beta_i$ each of which covers $\beta$. Each $\beta_i$ has the same number of sides as $\beta$. The group of decktransformations of $U_p$ transforms $H^*$ into itself, and carries $\beta_i$ into $\beta_j$, $i \neq j$. Hence $\beta_i$ together with a suitably chosen subset of the subgraph $F(\beta_i)$ of $H^*$ which bounds $\beta_i$ is a proper fundamental domain of the group of decktransformations of $U_p$. Hence the problem of determining the number of sides a proper fundamental domain of $U_p$ can have is equivalent to the problem of determining the number of sides $\beta$ can have for all possible choices of $H$.

*Lemma* 6. *Let $G$ be a connected graph containing no free arcs or chains, which is mapped on a torus $T$ in such a manner that $T - G$ is a single 2-cell $\alpha$. Then $\alpha$ has either four or six sides.*

Proof: $G$ contains at least one circuit $\lambda$, for otherwise $G$ would be a tree, but then $T - G$ cannot be a 2-cell. No circuit of $G$ separates $T$, for if it did, $T - G$ would not be connected, contrary to hypothesis. $G$ contains a circuit $\mu$ distinct from $\lambda$, for if $\lambda$ were the only circuit in $G$, $T - \lambda$ would be a cylinder which $G - \lambda$ does not separate, and then $T - G$ would not be a 2-cell. $\lambda$ meets $\mu$, for otherwise $T - (\lambda + \mu)$ would consist of two cylinders, and $T - G$ would not be connected.

$\lambda \cap \mu$ is connected, for suppose it were not. Cut $T$ along $\lambda$, $T$ will open out into a cylinder bounded by two simple closed curves $\lambda_1$, $\lambda_2$, each isomorphic with $\lambda$. $\mu - (\lambda \cap \mu)$ consists of at least two chains, each mapped in this cylinder, with ends on the boundary of the cylinder. If $\eta$, one of these chains, has both its ends on $\lambda_1$, or both its ends on $\lambda_2$, then $\eta$ separates the cylinder. On the other hand if each of these chains has one end on $\lambda_1$, and the other end on $\lambda_2$, then these chains together separate the cylinder. In either case $T - G$ is not connected, contrary to hypothesis.

$\lambda \cap \mu$ is either a single vertex of degree four, or an arc with endpoints of degree three. In the former case $G$ has two arcs, and if $T - G$

were cut along $G$ it would open out into a 2-cell of four sides, (each of the arcs of $G$ forming two sides), i.e. $T - (\lambda + \mu)$ is a 2-cell of four sides; in the latter case $G$ has three arcs, and $T - (\lambda + \mu)$ is a 2-cell of six sides.

Let $G' = \lambda + \mu$. Then $G'$ is identical with $G$. For suppose $G - G' \neq 0$. $G - G'$ cannot contain a circuit, for such a circuit, being mapped in the 2-cell $T - G'$, would separate it, but $T - G$ is connected. $G - G'$ contains no free arcs or chains, for if it did, $G$ would contain free arcs or chains, contrary to hypothesis. Hence $G - G'$ must contain a chain $\zeta$ having both its ends on $G'$, but then $\zeta$ separates $T - G'$, which contradicts the hypothesis that $T - G$ is connected. Hence $G - G' = 0$.

*Corollary. If $\mathfrak{D}$ is a proper fundamental domain of the group of decktransformations of the universal covering surface of a torus, $\mathfrak{D}$ has either four or six sides.*

Since any doubly infinite group $\mathfrak{G}$ of translations of the Euclidean plane into itself can be regarded as a representation of the group of decktransformations of the universal covering surface of a torus, it follows that a proper fundamental domain of the group $\mathfrak{G}$ has either four or six sides.

For example, let $\varphi(z)$ be a doubly periodic function of the complex variable $z$. By a fundamental domain of $\varphi(z)$ we mean a proper fundamental domain of the group $\mathfrak{G}$ of translations $\tau_{hk} : z' = z + h\omega_1 + k\omega_2$, where $\omega_1$, $\omega_2$, are the periods of $\varphi(z)$. It then follows that:

*A fundamental domain of a doubly periodic function of a complex variable has either four or six sides.*

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# A NEW APPROACH TO KRON'S WORK

By Nathaniel Coburn

I. **Introduction.** We propose to re-examine Kron's work, his problem, and give another approach to these matters.

II. **Notation.** We shall use the general notation of Schouten-Struik.[1] Consider a holonomic coördinate system $(x^\lambda)$ in a Riemannian space of $n$-dimensions $(V_n)$. The metric tensor and connection of $V_n$ are denoted, respectively by

$$a_{\lambda\mu}, \qquad (\alpha, \lambda, \mu = 1 \cdots n), \tag{1}$$

$$\Gamma^\lambda_{\alpha\beta} = \frac{a^{\lambda\nu}}{2}\left(\frac{\partial a_{\alpha\nu}}{\partial x^\beta} + \frac{\partial a_{\beta\nu}}{\partial x^\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x^\nu}\right). \tag{2}$$

For a holonomic coördinate system $(\bar{x}^{\bar{\lambda}})$ in an $n$-dimensional $(L_n)$, we write for the metric tensor and connection,

$$\bar{a}_{\bar{\lambda}\bar{\mu}}, \qquad (\bar{\alpha}, \bar{\lambda}, \bar{\mu} = 1 \cdots n), \tag{3}$$

$$\bar{\Gamma}^{\bar{\lambda}}_{\bar{\alpha}\bar{\beta}}. \tag{4}$$

III. **Machine of Type I.** The Lagrange equations of Kron's representative machine (I)[2] may be written as follows:

$$e_\alpha = R_{\alpha\beta} i^\beta + a_{\alpha\beta}\left(\frac{di^\beta}{dt} + \Gamma^\beta_{\lambda\mu} i^\lambda i^\mu\right). \tag{1}$$

Or using absolute differentiation,

$$e_\alpha = R_{\alpha\beta} i^\beta + a_{\alpha\beta}\frac{\delta i^\beta}{dt}. \tag{2}$$

The solutions of these differential equations will be said to determine the "paths" of $V_n$. Each path corresponds, of course, to a performance of the machine. Let the inductances in the machine of type (I) remain unaltered but let the resistances $(R_{\alpha\beta})$ and the impressed

[1] J. A. Schouten and D. J. Struik "Einführung in die Neueren Methoden der Differentialgeometrie," Gronigen, 1935.

[2] G. Kron "Non-Riemannian Dynamics of Rotating Electrical Machinery" Journal of Math. and Physics, Vol. XIII, No. 2, pg. 165.

voltages $(e_\alpha)$ be arbitrary. Then for each value of vector $(e_\alpha)$ and each tensor $(R_{\alpha\beta})$, we have a path in $V_n$; the various $(e_\alpha)$ and $(R_{\alpha\beta})$ determine the $\infty^{n-1}$ paths in $V_n$. It is to be noted that the $(e_\alpha)$ and $(R_{\alpha\beta})$ are $(2n)^2$ independent parameters which determine the paths. We note that at time $(t = 0)$, the charges $(x^\lambda, \lambda = 1 \cdots n-1)$ and rotor angle $(\theta = x^n)$ have the value zero. This means that all paths pass through the origin; the space $V_n$ is centered. A classification of $(e_\alpha)$ and $(R_{\alpha\beta})$ should determine the various performances of a particular machine $(a_{\lambda\mu}$ fixed) of type I.

Though the $(e_\alpha)$ and $(R_{\alpha\beta})$ fix the paths for machine of type I, the quantities $a_{\lambda\mu}$, which are functions of the inductances, determine the $V_n$. When the $a_{\lambda\mu}$ are altered by a tensor transformation (a coördinate transformation, implied) then $V_n$ is unaltered. Let us briefly study the transformations which take $(a_{\lambda\mu})$ of type I into $(a_{\lambda'\mu'})$ of type I. Such a study will tell us physically all values of the inductances which give a machine of type I (maybe taps); geometrically, what the group of $V_n$ is. For a machine of type I, the $a_{\lambda\mu}$ are given by the following matrix:[4]

(3) $a_{\lambda\mu} =$

| $\lambda$ \ $\mu$ | $d_s$ | $a$ | $b$ | $q_s$ | $t$ |
|---|---|---|---|---|---|
| $d_s$ | $L_{sd}$ | $M_d \cos\theta$ | $-M_d \sin\theta$ | 0 | 0 |
| $a$ | $M_d \cos\theta$ | $L_1 - L_2 \cos 2\theta$ | $L_2 \sin 2\theta$ | $M_q \sin\theta$ | 0 |
| $b$ | $-M_d \sin\theta$ | $L_2 \sin 2\theta$ | $L_1 + L_2 \cos 2\theta$ | $M_q \cos\theta$ | 0 |
| $q_s$ | 0 | $M_q \sin\theta$ | $M_q \cos\theta$ | $L_{sq}$ | 0 |
| $t$ | 0 | 0 | 0 | 0 | $L_{tt}$ |

Evidently, a general change in the components of this matrix will change the machine of type I into some other type; to our present knowledge, only the $M$'s and $L$'s are arbitrary. Hence for a machine of type I to go into a machine of type I, the new $(a_{\lambda'\mu'})$ must be multiples of the old $(a_{\lambda\mu})$[5]

[3] Kron "Non-Riemannian Dynamics" pg. 115.

[4] Kron "Non-Riemannian Dynamics" pg. 117.

[5] It is a subject for electrical investigation whether machines of type I may have $a_{\lambda\mu}$ of other than matrix (3). That is, can the general

$$a_{\lambda'\mu'} = \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^\mu}{\partial x^{\mu'}} a_{\lambda\mu}$$

be interpreted as machine I (with charges, etc.), i.e. does the given machine I constitute a special coördinate system on $V_n$.

$$a_{\lambda'\mu'} = A^{\lambda}_{\lambda'} A^{\mu}_{\mu'} a_{\lambda\mu}, \tag{4}$$

$$A^{\lambda}_{\lambda'} = 0 \quad \text{when} \quad \lambda \neq \lambda', \tag{5}$$

$$\text{i.e. } A^{1}_{2'}, A^{1}_{3'} \text{ etc. are zero.} \tag{6}$$

Since

$$A^{\lambda}_{\lambda'} = \frac{\partial x^{\lambda}}{\partial x^{\lambda'}}, \tag{7}$$

$$\frac{\partial A^{\lambda}_{\lambda'}}{\partial x^{\mu'}} = 0, \qquad \lambda' \neq \mu', \tag{8}$$

$$\begin{aligned} A^{1}_{1'} &= f^{1}(x^{1'}), \\ A^{2}_{2'} &= f^{2}(x^{2'}), \text{ etc.} \end{aligned} \tag{9}$$

Upon integration (8) becomes

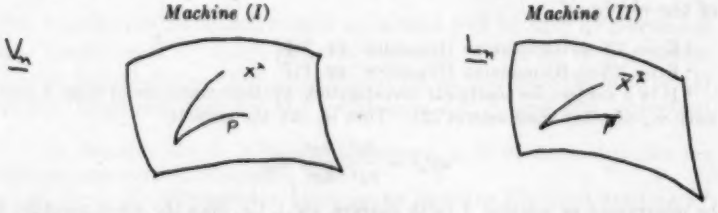
$$\begin{aligned} x^{1} &= g^{1}(x^{1'}), \\ x^{2} &= g^{2}(x^{2'}), \text{ etc.} \end{aligned} \tag{10}$$

In view of the fact that the charges do not enter the $a_{\lambda\mu}$ of machine I (as yet), and that $L_{ii}$ is independent of $(\theta)$, the $A^{\lambda}_{\lambda'}$ are constants and (10) becomes

$$\begin{aligned} x^{1} &= c^{1}x^{1'}, \\ x^{2} &= c^{2}x^{2'}, \text{ etc.} \end{aligned} \tag{11}$$

The equations (11) define the group of $V_n$—though this group will be enlarged by an investigation of all machines of type I.

**IV. Related Spaces—Machines of Type I and Type II.** We have represented the performance of machine I by the paths of an $X_n$ which is a $V_n$. Evidently the performance of machine (II) may be represented by the paths of an $X_n$ which is an $L_n$[6]—physically, the two machines differ in performance, not in physical nature. Let us make a comparative study




[6] Schouten and Struik "Einführung" pg. 75.

For the equations of paths in the machines, we have

$$(2) \qquad \frac{\delta i^\lambda}{dt} = e^\lambda + R^\lambda_{\cdot\beta} i^\beta \;\Big|\; \frac{\bar{\delta} \bar{i}^{\bar{\lambda}}}{dt} = \bar{e}^{\bar{\lambda}} + \bar{R}^{\bar{\lambda}}_{\cdot\bar{\beta}} \bar{i}^{\bar{\beta}}.$$

Let us consider the integral equations of (2),

$$(3) \qquad i^\lambda = g^\lambda(e^\alpha, R^\alpha_{\cdot\beta}, t) \;\Big|\; \bar{i}^{\bar{\lambda}} = g^{\bar{\lambda}}(\bar{e}^{\bar{\alpha}}, \bar{R}^{\bar{\alpha}}_{\cdot\bar{\beta}}, t)$$

$$(4) \qquad x^\lambda = f^\lambda(e^\alpha, R^\alpha_{\cdot\beta}, t) \;\Big|\; \bar{x}^{\bar{\lambda}} = f^{\bar{\lambda}}(e^{\bar{\alpha}}, \bar{R}^{\bar{\alpha}}_{\cdot\bar{\beta}}, t).$$

We notice that if the parameters $(e^\alpha, R^\alpha_{\cdot\beta})$ and $(\bar{e}^{\bar{\alpha}}, \bar{R}^{\bar{\alpha}}_{\cdot\bar{\beta}})$ could be put into correspondence, then a point correspondence (*along paths*) could be obtained between $V_n$ and $L_n$ (not necessarily a unique point to point *transformation*). Kron has given the following method of obtaining a correspondence of parameters.[7] Physically, let us fix the time as $(t = t_1)$; assume that the same amount of power is now supplied to both machines; assume that the magnetic field has the same energy in both machines;[8] that the amount of energy being dissipated is the same. These assumptions can be written as:

$$(5) \qquad P = e_\lambda i^\lambda = \bar{e}_{\bar{\lambda}} \bar{i}^{\bar{\lambda}},$$

$$(6) \qquad L = \tfrac{1}{2} a_{\lambda\mu} i^\lambda i^\mu = \tfrac{1}{2} \bar{a}_{\bar{\lambda}\bar{\mu}} \bar{i}^{\bar{\lambda}} \bar{i}^{\bar{\mu}},$$

$$(7) \qquad F = \tfrac{1}{2} R_{\lambda\mu} i^\lambda i^\mu = \tfrac{1}{2} \bar{R}_{\bar{\lambda}\bar{\mu}} \bar{i}^{\bar{\lambda}} \bar{i}^{\bar{\mu}},$$

$$(8) \qquad \bar{i}^{\bar{\lambda}} = C^{\bar{\lambda}}_{\mu} i^\mu.$$

The transformation (8) taken with (5), (6), (7) gives

$$(9) \qquad \bar{e}_{\bar{\lambda}} = C^\lambda_{\bar{\lambda}} e_\lambda,$$

$$(10) \qquad \bar{R}_{\bar{\lambda}\bar{\mu}} = C^{\lambda\mu}_{\bar{\lambda}\bar{\mu}} R_{\lambda\mu},$$

$$(11) \qquad \bar{a}_{\bar{\lambda}\bar{\mu}} = C^{\lambda\mu}_{\bar{\lambda}\bar{\mu}} a_{\lambda\mu}.$$

We decide that our correspondence of points $(P, \bar{P})$ and paths shall be determined by (9) and (10). When these relations are substituted into (4), then the paths of $V_n$ and $L_n$ are put into correspondence; when $(t)$ is given the same value in both expressions (4), then a point correspondence $(P \leftrightarrow \bar{P})$ is determined. In order to complete our point correspondence, we require that every *affinor field* $(w_{\lambda\mu}..)$ at $P$ be transferred into $(\bar{w}_{\bar{\lambda}\bar{\mu}}..)$ at $\bar{P}$,

$$(12) \qquad \bar{w}_{\bar{\lambda}\bar{\mu}}.. = C^{\lambda\mu\cdots}_{\bar{\lambda}\bar{\mu}\cdots} w_{\lambda\mu}....$$

[7] Kron "Non-Riemannian Dynamics" pg. 136.

[8] Same as 7.

This last step gives us the connection $\bar{\Gamma}^{\bar{\lambda}}_{\bar{\alpha}\bar{\beta}}$ of $L_n$. For

$$\nabla_{\bar{\lambda}} \bar{a}^{\bar{\mu}\bar{\nu}} = C^{\bar{\lambda}\bar{\alpha}\bar{\beta}}_{\lambda\mu\nu} \nabla_{\lambda} a^{\mu\nu}, \tag{13}$$

$$= 0,$$

$$Q^{\cdot\cdot\bar{\mu}\bar{\nu}}_{\bar{\lambda}} = \nabla_{\bar{\lambda}} \bar{a}^{\bar{\mu}\bar{\nu}} = 0. \tag{14}$$

Also

$$\nabla_{\bar{\lambda}} \bar{v}^{\bar{\mu}} = C^{\bar{\lambda}\bar{\mu}}_{\lambda\mu} \nabla_{\lambda} v^{\mu}, \tag{15}$$

$$= C^{\bar{\lambda}\bar{\mu}}_{\lambda\mu} (\partial_{\lambda} v^{\mu} + \Gamma^{\mu}_{\alpha\lambda} v^{\alpha}),$$

$$= \partial_{\bar{\lambda}} v^{\bar{\mu}} + C^{\bar{\alpha}\bar{\lambda}\bar{\mu}}_{\bar{\alpha}\lambda\mu} \Gamma^{\mu}_{\alpha\lambda} v^{\bar{\alpha}} + v^{\bar{\alpha}} C^{\bar{\mu}}_{\bar{\beta}} \partial_{\bar{\lambda}} C^{\bar{\beta}}_{\bar{\alpha}}.$$

Thus

$$\bar{\Gamma}^{\bar{\lambda}}_{\bar{\mu}\bar{\alpha}} = C^{\mu\alpha\bar{\lambda}}_{\bar{\mu}\bar{\alpha}\lambda} \Gamma^{\lambda}_{\mu\alpha} - C^{\mu\alpha}_{\bar{\mu}\bar{\alpha}} \partial_{\alpha} C^{\bar{\lambda}}_{\mu}, \tag{16}$$

$$\bar{S}^{\cdot\cdot\bar{\lambda}}_{\bar{\mu}\bar{\alpha}} = \bar{\Gamma}^{\bar{\lambda}}_{[\bar{\mu}\bar{\alpha}]} = -C^{\mu\alpha}_{\bar{\mu}\bar{\alpha}} \partial_{[\alpha} C^{\bar{\lambda}}_{\mu]}. \tag{17}$$

From the work of Schouten[9]

$$\bar{\Gamma}^{\bar{\lambda}}_{\bar{\mu}\bar{\alpha}} = \tfrac{1}{2} \bar{a}^{\bar{\lambda}\bar{\rho}} (\partial_{\bar{\alpha}} \bar{a}_{\bar{\mu}\bar{\rho}} + \partial_{\bar{\mu}} \bar{a}_{\bar{\rho}\bar{\alpha}} - \partial_{\bar{\rho}} \bar{a}_{\bar{\alpha}\bar{\mu}}) \tag{18}$$

$$+ \bar{S}^{\cdot\cdot\bar{\lambda}}_{\bar{\mu}\bar{\alpha}} - \bar{a}^{\bar{\lambda}\bar{\beta}} (\bar{a}_{\bar{\mu}\bar{\rho}} \bar{S}^{\cdot\cdot\bar{\rho}}_{\bar{\beta}\bar{\alpha}} + \bar{a}_{\bar{\alpha}\bar{\rho}} \bar{S}^{\cdot\cdot\bar{\rho}}_{\bar{\beta}\bar{\mu}}).$$

It is to be noted that our approach (5, 6, 7, 8) differs from Kron's in that he uses $(C^{\bar{\lambda}}_{\mu})$ as a coördinate transformation affinor; we use for transferring vector *fields* from $V_n$ to $L_n$ (and vice versa).

V. **The Path Transformation Affinor** $(B^{\bar{\lambda}}_{\lambda})$. Since in the actual operation of machines (I) and (II), we cannot at any particular time satisfy the conditions for the validity of (IV, 5, 6, 7), hence (IV, 8) is not valid. Another way of looking at the matter is to say that from (IV, 5, 6, 7) we can establish a *correspondence of paths* and vector fields. But we cannot determine what the correspondence of *tangent vectors along paths* in $V_n$ and $L_n$ shall be. By solving one set of equations of (IV, 4) for $(t)$ and substituting into the other set of equations of (IV, 4), we get

$$\bar{x}^{\bar{\lambda}} = \bar{y}^{\bar{\lambda}}(x^{\alpha}, e^{\alpha}, R^{\alpha\beta}, \bar{e}^{\bar{\alpha}}, \bar{R}^{\bar{\alpha}\bar{\beta}}). \tag{1}$$

Along a path of $V_n$ and the corresponding path of $L_n$ determined by

$$\bar{e}_{\bar{\lambda}} = C^{\lambda}_{\bar{\lambda}} e_{\lambda}, \tag{2}$$

[9] Schouten and Struik "Einführung" pg. 83.

$$\bar{R}_{\bar{\lambda}\bar{\mu}} = C^{\lambda\mu}_{\bar{\lambda}\bar{\mu}} R_{\lambda\mu}, \tag{3}$$

the quantities $(e^\alpha, R^{\alpha\beta}, \bar{e}^{\bar{\alpha}}, \bar{R}^{\bar{\alpha}\bar{\beta}})$ involve constants (along path) and functions of $(x^n \equiv \theta)$. If we substitute (2), (3) into (1), we get

$$\bar{x}^{\bar{\lambda}} = f^{\bar{\lambda}}(x^\lambda, e^\lambda, R^{\lambda\mu}). \tag{4}$$

It must be remembered that in (4) although the $(e^\lambda, R^{\lambda\mu})$ occur as explicit and arbitrary numbers, a specific set of these numbers is associated with every set $(x^\lambda, i^\lambda)$ of a path. We must write

$$\begin{aligned} e^\lambda &= g^\lambda(x^\alpha, i^\alpha), \\ R^{\lambda\beta} &= G^{\lambda\beta}(x^\alpha, i^\alpha). \end{aligned} \tag{5}$$

Another method of saying this, is the following: Paths may cross, at such a cross-point, we have two sets of values of $(e^\lambda, R^{\lambda\mu})$. Hence $(e^\lambda, R^{\lambda\mu})$ cannot depend on $(x^\lambda)$ alone; they depend on the direction of the path at the point. A cross-point can be physically realized as follows: consider machine I operating with $\left(e^\alpha_1, R^{\alpha\beta}_1\right)$

$$x^\lambda = f^\lambda\left(e^\alpha_1, R^{\alpha\beta}_1, t\right) \tag{6}$$

until time $\left(t = t_1\right)$ when suddenly $\left(e^\alpha_2, R^{\alpha\beta}_2\right)$ replace $\left(e^\alpha_1, R^{\alpha\beta}_1\right)$.

From the manner in which (4) was derived:

$$\frac{d\bar{x}^{\bar{\lambda}}}{dt} = \frac{\partial \bar{x}^{\bar{\lambda}}}{\partial x^\lambda} \frac{dx^\lambda}{dt}, \tag{7}$$

$$B^{\bar{\lambda}}_\lambda = \frac{\partial \bar{x}^{\bar{\lambda}}}{\partial x^\lambda}, \tag{8}$$

$$\bar{i}^{\bar{\lambda}} = B^{\bar{\lambda}}_\lambda i^\lambda. \tag{9}$$

We can now explicitly state Kron's problem. *Kron's Problem is to find $B^{\bar{\lambda}}_\lambda$, given $V_n$ and $L_n$.*

**VI. Kron's Problem and Other Problems.** Before indicating how Kron's problem may be solved, let us mention some of the other fundamental problems in the theory.

*Problem:* to find the group of $V_n$ and $L_n$. In other words, what are the most general transformations $[x^{\lambda'} = x^{\lambda'}(x^\mu)]$ taking machine I (without taps, etc.) into another machine I (with taps, etc.)? Can every transformation $[x^{\lambda'} = x^{\lambda'}(x^\mu)]$ on machine I be interpreted as taking a machine I into another machine I?

We give the following method of attacking Kron's problem. Let a coördinate transformation be induced by (V, 4) in $L_n$,

$$\bar{x}^\lambda = f^\lambda(\bar{x}^\lambda, e^a, R^{ab}). \tag{1}$$

Then in the new coördinate system $(\bar{x}^\lambda)$, the tangent vectors to the paths are given by

$$\frac{d\bar{x}^\lambda}{dt}\Big)_{L_n} = B^\lambda_\lambda \frac{dx^\lambda}{dt} = \frac{dx^\lambda}{dt}\Big)_{V_n}. \tag{2}$$

That is the path equations have the same solutions in $(\bar{x}^\lambda)$ of $L_n$ as in $(x^\lambda)$ of $V_n$. Before proceeding further in finding $(B^\lambda_\lambda)$, we must point out that $(B^\lambda_\lambda)$ must first satisfy a condition obtained previously. From

$$\nabla_\mu \bar{a}_{\lambda\nu} = 0, \tag{3}$$

we have

$$\frac{\partial \bar{a}_{\lambda\nu}}{\partial x^\mu} - \bar{\Gamma}^\alpha_{\nu\mu}\bar{a}_{\lambda\alpha} - \bar{\Gamma}^\alpha_{\lambda\mu}\bar{a}_{\nu\alpha} = 0. \tag{4}$$

Here $(\bar{a}_{\lambda\mu})$ are known as functions of $(x^\lambda)$, from (IV, 13). Likewise $(\bar{\Gamma}^\alpha_{\nu\mu})$ are known (IV, 16). Thus we have the condition on $(B^\mu_\lambda)$

$$B^\mu_\mu \frac{\partial \bar{a}_{\lambda\nu}}{\partial x^\mu} - \bar{\Gamma}^\alpha_{\nu\mu}\bar{a}_{\lambda\alpha} - \bar{\Gamma}^\alpha_{\lambda\mu}\bar{a}_{\nu\alpha} = 0. \tag{5}$$

There are $\frac{n^2(n+1)}{2}$ equations in the $n^2$ unknowns $(B^\mu_\lambda)$. Generally, this would overdetermine the $(B^\mu_\lambda)$. But the $(\bar{a}_{\lambda\nu})$ are functions of only one variable $(x^n)$. Hence at most only $(B^\mu_\lambda)$ are determined. In order to find the $(B^\mu_\lambda)$, we must study the paths of $V_n$ and $L_n$. From (2), we see that our problem is similar in some details to the problem of spaces with corresponding geodesics.[10] We shall denote the quantities of $L_n$ in $(\bar{x}^\lambda)$ by $(\bar{a}_{\lambda\mu}, \cdots)$. Thus the paths in $L_n$ and $V_n$ become

$$\frac{d\bar{v}^\nu}{dt} + \bar{\Gamma}^\nu_{\lambda\mu}\bar{v}^\lambda\bar{v}^\mu = \bar{e}^\nu + \bar{R}^\nu_{.\lambda}\bar{v}^\lambda, \tag{6}$$

$$\frac{dv^\nu}{dt} + \Gamma^\nu_{\lambda\mu}v^\lambda v^\mu = e^\nu + R^\nu_{.\lambda}v^\lambda. \tag{7}$$

Thus

$$(\bar{\Gamma}^\nu_{(\lambda\mu)} - \Gamma^\nu_{\lambda\mu})\bar{v}^\lambda\bar{v}^\mu = (\bar{e}^\nu - e^\nu) + \bar{v}^\lambda(\bar{R}^\nu_{.\lambda} - R^\nu_{.\lambda}). \tag{8}$$

[10] L. P. Eisenhart "Riemannian Geometry," Princeton, 1926, pg. 131.

At any point $(x^\lambda)$ for every value of $(i^\lambda)$, we find distinct values for $(e^\lambda, R^{\lambda\mu}i_\mu)$. In order to find what *vector* relations exist between $(e^\lambda, R^{\lambda\mu}i_\mu)$, $('e^\lambda, R^{\lambda\mu}i_\mu)$ and the vector $(i^\lambda)$, we argue as follows: assume that $e^\lambda$ and $i^\lambda$ are inclined at some angle other than zero degrees at $P$,

then we may change $i^\lambda$ to $'i^\lambda$ (some position in the above cone of directions about $e^\lambda$ at $P$). Now $(e^\lambda)$ depends only on $(i^\lambda)$; hence only on the magnitude of $(i^\lambda)$ and the angle between $(i^\lambda)$ and $(e^\lambda)$. Similarly $('e^\lambda)$ depends on the same function of $('i^\lambda)$ (a function involving only the magnitude of $'i^\lambda$ and the angle $\phi$). Let the magnitude of $'i^\lambda$ be the equal to that of $i^\lambda$. Since $'i^\lambda$ lies in the above cone, the angle $('\phi = \phi)$. Hence $'e^\lambda = e^\lambda$. But this means that different paths at $P$ are determined by the same $e^\lambda$ (and $R^{\lambda\mu}$). This is impossible. Hence $(\phi = 0)$

$$e^\lambda = \sigma i^\lambda. \tag{9}$$

The same argument may be repeated for the vector $(R^{\lambda\mu}e_\lambda)$. Hence

$$R^{\lambda\mu}e_\lambda = \bar{\rho}i^\lambda. \tag{10}$$

But

$$R^{\lambda\mu}e_\lambda = \sigma R^{\lambda\mu}i_\lambda. \tag{11}$$

Hence

$$R^{\lambda\mu}i_\lambda = \rho i^\mu. \tag{12}$$

Thus equation (8) becomes

$$(\bar{\Gamma}^\nu_{(\lambda\mu)} - \Gamma^\nu_{\lambda\mu})i^\lambda i^\mu = \sigma i^\nu, \tag{13}$$

where $\sigma$ is a function of the coördinates of the point under consideration. Since (13) is valid for all $(i^\lambda)$, from the work of Schouten,[11] we have

$$\bar{\Gamma}^\nu_{(\lambda\mu)} - \Gamma^\nu_{\lambda\mu} = A^\nu_\lambda p_\mu + A^\nu_\mu p_\lambda, \tag{14}$$

[11] J. A. Schouten "Der Ricci-Kalkül," Berlin, 1924, pg. 76.

where $(p_\lambda)$ is a vector—at present undetermined. The equations (14) constitute the "path conditions" on the $(B^\alpha_{\hat\beta})$. For $(\bar\Gamma^\nu_{(\lambda\mu)})$ are given in terms of the known $\bar\Gamma^{\hat\rho}_{(\hat\lambda\hat\beta)}$ by means of[12]

$$\bar\Gamma^{\hat\rho}_{(\hat\lambda\hat\beta)} B^\nu_{\hat\rho} = B^{\hat\lambda}_\lambda B^\alpha_{\hat\beta} \bar\Gamma^\nu_{(\lambda\alpha)} + \frac{\partial B^\nu_{(\hat\lambda}}{\partial x^\alpha} B^\alpha_{\hat\beta)}. \tag{15}$$

When (15) are substituted into (14), we obtain a new set of conditions on $(B^{\hat\lambda}_\lambda)$. Before giving these conditions in their final form, let us study the vector $(p_\lambda)$. From (IV, 18), we find

$$\bar\Gamma^\nu_{(\lambda\mu)} = \overline{\left\{ {\nu \atop \lambda\mu} \right\}} - 2S^{\cdot\cdot\alpha}_{\beta(\lambda}\, a_{\mu)\alpha} a^{\nu\beta}. \tag{16}$$

Thus

$$\overline{\left\{ {\nu \atop \lambda\mu} \right\}} - 2S^{\cdot\cdot\alpha}_{\beta(\lambda}\, a_{\mu)\alpha} a^{\nu\beta} - \left\{ {\nu \atop \lambda\mu} \right\} = A^\nu_\lambda p_\mu + A^\nu_\mu p_\lambda. \tag{17}$$

Contracting on $(\nu, \mu)$, (17) becomes

$$\overline{\left\{ {\nu \atop \lambda\nu} \right\}} - S^{\cdot\cdot\nu}_{\nu\lambda} - \left\{ {\nu \atop \lambda\nu} \right\} = (n+1)p_\lambda. \tag{18}$$

Thus $(p_\lambda)$ also involves functions of $(B^\alpha_{\hat\beta})$. We shall study this briefly. It is easily shown that[13]

$$\overline{\left\{ {\nu \atop \lambda\nu} \right\}} = \frac{\partial \log \sqrt{\bar a}}{\partial x^\lambda}, \tag{19}$$

where

$$\bar a = |\, \bar a_{\lambda\mu} \,|. \tag{20}$$

Now since

$$\bar a_{\lambda\mu} = B^{\hat\lambda}_\lambda B^{\hat\mu}_\mu \bar a_{\hat\lambda\hat\mu}, \tag{21}$$

$$\bar a = B^2 a(x^{\hat\lambda}), \tag{22}$$

$$B = |\, B^{\hat\lambda}_\lambda \,|. \tag{23}$$

Thus (18) becomes

$$\frac{\partial \log (B\sqrt{a(x^{\hat\lambda})}}{\partial x^\lambda} - S^{\cdot\cdot\nu}_{\lambda\nu} - \frac{\partial \log \sqrt{a}}{\partial x^\lambda} = (n+1)p_\lambda. \tag{24}$$

[12] Schouten "Ricci Kalkül" pg. 65.
[13] Eisenhart "Riemannian Geometry" pg. 18.

Thus the vector $(p_\lambda)$ involves known functions and the unknown determinant $(B)$. We can simplify (24) by remembering that

$$\bar{a}_{\bar{\lambda}\bar{\mu}} = C^{\lambda\mu}_{\bar{\lambda}\bar{\mu}} a_{\lambda\mu}; \tag{25}$$

$$\bar{a}(x^{\bar{\lambda}}) = C^2 a. \tag{26}$$

By the work of Kron[14]

$$C = |C^{\bar{\lambda}}_{\lambda}| = 1. \tag{27}$$

Thus (24) becomes

$$\frac{\partial \log B}{\partial x^\lambda} - \overset{\cdot\cdot\nu}{S}_{\nu\lambda} = (n+1)p_\lambda. \tag{28}$$

It would be rather difficult to solve the system (28), (15), (14). Hence we shall *assume that* $B = 1$. That is, that the group of $L_n$ is equivolumar (also $V_n$). This hypothesis must be analyzed for physical significance. It may furnish a solution to the problem mentioned at the beginning of VI. Thus (28) becomes

$$-\overset{\cdot\cdot\nu}{S}_{\nu\lambda} = (n+1)p_\lambda. \tag{29}$$

Hence our general theorem is:

The affinor $(B^\alpha_\beta)$ which takes paths of $V_n$ into paths of $L_n$ is completely determined by the conditions

$$\nabla_{\bar{\mu}} \bar{a}_{\bar{\lambda}\bar{\nu}} = 0,$$

$$\bar{\Gamma}^\nu_{(\lambda\mu)} - \Gamma^\nu_{\lambda\mu} = A^\nu_\lambda p_\mu + A^\nu_\mu p_\lambda,$$

where

$$B^\nu_{\bar{\rho}} \bar{\Gamma}^{\bar{\rho}}_{(\bar{\lambda}\bar{\mu})} = B^{\bar{\lambda}}_\lambda B^\mu_{\bar{\mu}} \Gamma^\nu_{(\lambda\mu)} + \frac{\partial B^\nu_{(\bar{\lambda}}}{\partial x^\mu} B^\mu_{\bar{\mu})},$$

$$-\overset{\cdot\cdot\nu}{S}_{\nu\lambda} = (n+1)p_\lambda.$$

These equations involve the $(B^\alpha_{\bar{\beta}})$ and their partial derivatives. This completes our general study of Kron's problem. The significance of this work (as a solution of Kron's problem) depends on the fact that solutions of the above system exist. We leave this problem to the engineer.

VII. **Some New Problems in Geometry.** Aside from the possible utility of the above as a solution of Kron's problem, we can point out

[14] Kron "Non-Riemannian Dynamics" pg. 129.

some new problems using the ideas developed. We shall *reverse* the approach used above.

Consider:

(1) a metric $L_n(a_{\lambda\mu}, \Gamma^r_{\lambda\mu})$ and another metric $\bar{L}_n(\bar{a}_{\lambda\mu}, \bar{\Gamma}^r_{\lambda\mu})$. When can a field correspondence $(C^\lambda_\mu)$ be set up

(2)
$$\bar{v}_{\lambda\mu} = C^{\alpha\beta}_{\lambda\mu} w_{\alpha\beta},$$
$$w_{\alpha\beta} = \bar{C}^{\lambda\mu}_{\alpha\beta} \bar{v}_{\lambda\mu},$$

such that geodesics correspond (i.e.).

(3)
$$\bar{\Gamma}^r_{(\lambda\mu)} = \Gamma^r_{\lambda\mu} + A^r_\mu p_\lambda + A^r_\lambda p_\mu.$$

There are three types of questions that one can ask

(1) assumptions on $L_n$ ; $\bar{L}_n$ .
(2) conditions on $(C^\lambda_\mu)$
(3) conditions on the correspondence (i.e. paths into paths; same conformal geometry etc.).

The work of H. Schapiro of Moskau and A. Dushek is along these lines. We shall investigate this field in a future paper.

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# TRIGONOMETRIC INTERPOLATION OF EMPIRICAL AND ANALYTICAL FUNCTIONS

By C. Lanczos

The purpose of the following article is to show the manifold possibilities of the application of Tshebysheff's polynomials in approximation problems. All these applications are based on some simple basic properties of the Fourier series. Although the mathematical nature of the Fourier series is exhaustively investigated, it is not sufficiently realized how excellent approximations, for both empirical and analytical functions, may be obtained by combining the advantages of the power series and the Fourier series. The chief difficulty in putting the Fourier series into practise is frequently the fact that the integrals which determine formally the coefficients of the series, are eventually not adapted for actual evaluation. The present investigation shows how in a wide class of problems the evaluation of those integrals can be avoided and substituted by the easily accessible method of the trigonometric interpolation.

## I. Interpolating and Extrapolating Series

Two fundamentally different types of series expansions are widely used in pure and applied mathematics. Characteristic of the one type is the *Taylor series* which expands a given analytical function in the neighbourhood of a point into an infinite power series. Characteristic of the other type is the *Fourier series* which uses the trigonometric functions as members of the infinite series and includes a much wider class of functions since the analytical nature of the function is not required.

Similar to the Taylor series in nature are the so-called "asymptotic expansions" which have likewise the form of an infinite power series but their realm of convergence is zero. Similar to the Fourier series in nature are the various expansions which substitute the trigonometric functions by some other complete orthogonal function systems.

The present paper discusses the mutual relation of the different types of infinite series from the common platform of *interpolation problems*. We start our considerations with the fundamental interpolation formula



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of *Newton-Gregory*.[1] Given the values of the function $f(x)$ at the $n$ equidistant points

$$x = a,\ a + \epsilon,\ \cdots a + (n-1)\epsilon \tag{1.1}$$

called shortly the "points of interpolation." Then the following power series of the $n-1^{st}$ order fits the given function in the points of interpolation exactly and may be considered as an approximation of $f(x)$[2] anywhere between the two end-points $x = a$ and $x = a + (n-1)\epsilon$:

$$\overline{f(a+x)} = f(a) + \left(\frac{\Delta f}{\Delta x}\right)_{x=a}[x] + \frac{1}{2!}\left(\frac{\Delta^2 f}{\Delta x^2}\right)_{x=a}[x]^2 + \cdots \frac{1}{(n-1)!}\left(\frac{\Delta^{n-1} f}{\Delta x^{n-1}}\right)_{x=a}[x]^{n-1} \tag{1.2}$$

The notations are:

$$\Delta x = \epsilon \tag{1.3}$$

$$[x]^k = x(x-\epsilon)(x-2\epsilon)\cdots(x-(k-1)\epsilon) \tag{1.4}$$

1. **Taylor series and asymptotic expansions.** If $\epsilon$ decreases toward zero and $f(x)$ is analytical at the point $x = a$, the difference-coefficients of the series (1.2) converge in the limit $\epsilon = 0$ to the derivatives $f^{(k)}(a)$ while the functions $[x]^k$ converge to the powers $x^k$. This yields the formula

$$\overline{f(a+x)} = f(a) + f'(a)x + \cdots \frac{f^{(n-1)}(a)}{(n-1)!}x^{n-1} \tag{1.5}$$

This is the well known Taylor-expansion, limited to the first $n$ terms.

The infinite Taylor series can thus be considered as a successive approximation of the analytic function $f(x)$ by means of an interpolation process. We interpolate $f(x)$ in the neighborhood of $x = a$ in the $n$ equidistant points (1.1) the mutual distance of which decreases toward zero. We then increase the number $n$ indefinitely.

This process shows that the Taylor series is a typically *extra-polating* series. The realm of interpolation shrinks toward zero and we make use of the series *outside* of that realm. The fact that the series (1.5) converges with increasing $n$ indefinitely toward $f(x)$ in a finite realm of the complex plane, namely within the "radius of convergence," is a

[1] See e.g. *Whittaker-Robinson*, Ref. 13, p. 10; *Scarborough*, Ref. 10, p. 47.

[2] Throughout the paper $\overline{f(x)}$ means "approximation of $f(x)$."

well known result of the theory of analytical functions which is by no means self-evident, considering the fact that we obtain an approximation by interpolation and then use it *outside* of the original range.

We have indeed examples where the series (1.2), decreasing $\epsilon$ toward zero and then increasing $n$ toward infinity, does not converge at *any* point outside of the center of expansion $x = a$. This is the case with the so-called "asymptotic expansions." The point of expansion is here frequently the point $x = \infty$ and the series proceeds in *negative* powers of $x$. This, however, is merely a formal difference since the reciprocal transformation

$$z = \frac{1}{x} \tag{1.6}$$

transforms the point $a' = \infty$ into the origin $a = 0$ and the negative powers of $x'$ are transformed into positive powers of the new variable $z$.

The point $z = 0$ is now no longer an analytical point of the complex variable $z$ and thus the derivatives $f^{(k)}(a)$ do not exist in the complex sense. And still the interpolation formula (1.2) may converge to a definite limit as $\epsilon$ approaches zero. For that purpose the existence of $f^{(k)}(a)$ in the analytic sense is not required. It suffices that the

$$\lim_{\Delta x \to 0} \frac{\Delta^k f}{\Delta x^k} \tag{1.7}$$

shall exist *in that particular direction* of the complex plane which contains the points (1.1) of the interpolation.

This condition is indeed fulfilled in the case of "asymptotic expansions." Thus, although the customary definition of *Poincaré* pursues a different view point,[3] it seems to be justified (if not more adequate, as the author believes) to consider the asymptotic expansions, in complete similarity to the Taylor series, as the result of a *successive interpolation process*, fitting the function in $n$ infinitely close points which are lined up along one ray originating from the point $x = a$, and then increasing $n$ toward infinity. This view point shows the similarity of the asymptotic expansion with the Taylor series in the sense that they are both *extrapolating* series. The only difference is that in case of the Taylor series there exists a definite *finite* region within which the approximation converges monotonously towards the function $f(x)$ if $n$ increases toward infinity. Whereas in case of the

[3] See *Whittaker-Watson*, Ref. 12, p. 151.

asymptotic series the region within which the error decreases indefinitely with increasing $n$, shrinks towards zero as $n$ approaches infinity.

2. **Fourier series and other orthogonal expansions.** Quite different in nature are those series expansions which are based on the theory of orthogonal function-systems. Here we have an apriori given definite range of the real variable $x$, characterized by the limits $x = a$ and $x = b$. We want to represent the function $f(x)$ only *within these limits* and not outside of the given range. In fact the function need not to exist even outside of the range where it is defined. Hence these expansions are truly *interpolating* and not extrapolating series.

Due to the different character of the process of interpolation and extrapolation the "error" of the approximation $f(x)$, defined by

$$\eta(x) = f(x) - \overline{f(x)} \tag{1.8}$$

behaves in the two cases quite differently. The Taylor series and the asymptotic expansions extrapolate the function $f(x)$ outside of the point $x = a$, the neighbourhood of which determines the coefficients of expansion. Thus, taking a sufficiently large but finite number of both series, the error decreases to very small amounts if we are only near enough to the center of expansion. But the error increases rapidly as we move away from the center. Both series represent an "approximation in the small" or "approximation about a point." The distribution of errors is quite uneven if we consider any *finite* interval around the center of expansion.

Let us consider, on the other hand, an orthogonal expansion, confined from the very beginning to a definitely given finite range. There is no "center of expansion" here and no infinitesimal surroundings of one particular point. All points of the given range are equivalent and the whole range may be considered as the magnified image of the "infinitesimal surroundings" before. In no part of the range is the error now exceedingly small but, on the other hand, the error increases nowhere in its order of magnitude. We have here an "approximation in the large," with a practically *even* distribution of errors all over the given range.[4]

[4] For the present purposes we restrict ourselves to a limited class of functions of sufficiently regular behaviour. The function $f(x)$ is supposed to be finite and single valued, and continuous and differentiable everywhere within the given range, with a finite number of maxima and minima. We also assume that the number of approximating terms is sufficiently large.

## II. Interpolating approximation by means of orthogonal function-systems

Given a finite set of orthogonal functions

$$\varphi_0(x), \quad \varphi_1(x), \quad \varphi_2(x) \cdots \varphi_{n-1}(x) \tag{2.1}$$

belonging to the range

$$a \leqq x \leqq b \tag{2.2}$$

The function $f(x)$ which shall be restricted by the general regularity conditions enumerated in footnote 4, shall be approximated by the expansion

$$\overline{f(x)} = C_0\varphi_0(x) + C_1\varphi_1(x) + \cdots C_{n-1}\varphi_{n-1}(x) \tag{2.3}$$

The classical method to obtain a close approximation is based on the method of minimizing the "average error" $E$ defined by

$$E^2 = \int_a^b [f(x) - \overline{f(x)}]^2\,dx \tag{2.4}$$

which leads to the well known formula

$$C_k = \frac{\int_a^b f(x)\varphi_k(x)\,dx}{\int_a^b \varphi_k^2(x)\,dx} \tag{2.5}$$

In spite of the extraordinary importance of this method for theoretical investigations, the practical application of the method is frequently frustrated by the fact that the integrals

$$I_k = \int_a^b f(x)\varphi_k(x)\,dx \tag{2.6}$$

are practically not available. Even if the analytical law that controls $f(x)$ is known, the integrals (2.6) lead easily to forms of definite integrals, the evaluation of which is prohibitive. Then the cumbersome method of numerical integration has to be employed. The same holds if $f(x)$ is not given as an analytical function but only tabulated as an empirical function, in a finite number of ordinarily equidistant points. In both cases the integrals (2.6) can be obtained but approximately and usually after laborious calculations.

The method of determining the coefficients $C_k$ of the approximating

series discussed in the present paper needs no integration, but only the solution of a system of linear equations. Thus the determination of the coefficients is frequently much simpler than the classical method. These coefficients do not coincide exactly with the classical coefficients. But the complete series (2.3) yields practically the same accuracy, if $n$ is sufficiently large. If $n$ is too small for an efficient approximation, then also the classical method is unable to provide satisfactory results.

The general theory of orthogonal function systems[5] shows that the error of the approximation obtained by minimizing the average error has an *oscillatory* character: $\overline{f(x)}$ embraces the true function $f(x)$ in *waves* and the number of oscillations corresponds to the number $n$ of approximating functions. With increasing $n$ the number of oscillations increases permanently and the oscillatory nature of the error

$$\eta(x) = f(x) - \overline{f(x)} \tag{2.7}$$

remains preserved even for arbitrarily large $n$. The improvement of the approximation is due to the *amplitudes* of the oscillations which decrease monotonously toward zero. But still there are only $n$ points $x_k (k = 1, 2, \cdots n)$, where the error $\eta(x)$ vanishes exactly. If these points would be known, the coefficients of the approximation could be obtained merely by *interpolation*, solving the $n$ conditions

$$\overline{f(x_k)} = f(x_k) \qquad (k = 1, 2, \cdots n) \tag{2.8}$$

for the $n$ unknowns

$$C_0, C_1, \cdots C_{n-1} \tag{2.9}$$

Although the exact position of the points $x_k$ is not known, one can frequently *estimate* them closely enough to provide practically the full accuracy of the approximating series (2.3). We consider the *infinite* series:

$$f(x) = C_0\varphi_0(x) + C_1\varphi_1(x) + \cdots \tag{2.10}$$

obtained with the help of the *complete* function system

$$\varphi_0(x), \varphi_1(x), \cdots \tag{2.11}$$

Hence the error of the approximation (2.3) becomes:

$$\eta_n(x) = C_n\varphi_n(x) + C_{n+1}\varphi_{n+1}(x) + \cdots \tag{2.12}$$

[5] Instead of the original literature we quote here the classical reference book: *Courant-Hilbert*, Ref. 1, especially Chapters V and VI, which deal with that class of orthogonal function systems we have particularly in mind: function systems generated by a Sturm-Liouville problem.

If this series has a sufficiently rapid convergence, we may consider the first term of the series as the dominating term and neglect the higher terms. Thus we *estimate* $\eta_n(x)$ by putting

$$\overline{\eta_n(x)} = C_n \varphi_n(x) \tag{2.13}$$

and get the result that the error vanishes, i.e. function and approximation coincide, at the zeros of the function $\varphi_n(x)$. This means: *let the points of interpolation $x_k$ coincide with the zeros of the first neglected orthogonal function.*

The estimation (2.13) is not bad even if the series (2.12) converges slowly. It is true that then the series cannot be reduced to practically one term. But, provided that $n$ is sufficiently high, the zeros of $\varphi_{n+1}(x)$, $\varphi_{n+2}(x)$, $\cdots$ are in their majority near to the zeros of $\varphi_n(x)$ and thus the actual distribution of the zeros of $\eta_n(x)$ is not so essentially different from the zeros of $\varphi_n(x)$. Moreover, it is not our purpose to produce exactly the coefficients of the classical approximation. Even if the coefficients $c_k$, obtained by interpolation, are different from the classical coefficients $C_k$, obtained by integration, the resulting series may still compete successfully with the classical series, as far as closeness of the approximation is concerned. The integration method has the advantage that we get immediately the *final* values of $c_k$ which do not change if more and more orthogonal functions $\varphi_k(x)$ are taken in consideration. The interpolation method yields coefficients which are all the time changing with the number $n$ of approximating terms. In the case of the Fourier series—the only case to which the method will be applied on the following pages—an explicit relation can be established between the coefficients obtained by interpolation and those obtained by integration. One can see here that the coefficients obtained by interpolation converge toward the coefficients (2.5), which are the classical Fourier-coefficients, and in the limit $n = \infty$ become identical with them. One can also show that the interpolating series, as far as maximal error at any point of the range is concerned, is not behind the Fourier series of the same number of terms.[6]

## III. Special advantages of the trigonometric interpolation

Amongst all other orthogonal function-systems, the Fourier series is distinguished by some outstanding features which have particular

[6] Probably the same conditions hold for other orthogonal expansions provided that there is a "natural sequence" observed in the $\varphi_k(x)$ established by the condition that the number of zeros in the given interval for the successive $\varphi_k(x)$ coincides with the index $k = 0, 1, 2, \cdots$.

bearing also on the method of determining the coefficients of an approximation by interpolation. Due to these features, the "trigonometric interpolation" is in both its practical and theoretical aspects of particular interest and has its outstanding advantages compared with other interpolations. We shortly review these special properties of the Fourier series.[7]

The range of the Fourier series may be normalized to

$$a = -\pi, \qquad b = \pi \tag{3.1}$$

The sine and the cosine series can be separated by forming at first:

$$\begin{aligned} f_1(x) &= \tfrac{1}{2}[f(x) + f(-x)] \\ f_2(x) &= \tfrac{1}{2}[f(x) - f(-x)] \end{aligned} \tag{3.2}$$

so that

$$f(x) = f_1(x) + f_2(x) \tag{3.3}$$

The function $f_1(x)$ is an *even*, the function $f_2(x)$ an *odd* function:

$$\begin{aligned} f_1(x) &= \phantom{-}f_1(-x) \\ f_2(x) &= -f_2(-x) \end{aligned} \tag{3.4}$$

Thus the Fourier expansion of $f_1(x)$ will contain only the cosine terms, the expansion of $f_2(x)$ only the sine-terms. Hence we split the complete series into the two respective parts and reduce at the same time the range to the half range

$$a = 0, \qquad b = \pi \tag{3.5}$$

1. **Interpolation by sine-functions.** At first we consider the expansion of an *odd* function into a sine-series:

$$f(x) = a_1 \sin x + a_2 \sin 2x + \cdots \tag{3.6}$$

The orthogonal functions are here

$$\varphi_k(x) = \sin kx \tag{3.7}$$

The zeros of $\varphi_k(x)$ are:

$$x_\alpha = \alpha \frac{\pi}{k} \qquad (\alpha = 0, 1, 2, \cdots k) \tag{3.8}$$

[7] The method of trigonometric interpolation is usually treated in the text books of Practical Analysis; see Ref. 8, p. 211; Ref. 9, p. 126; Ref. 10, p. 388; Ref. 13, p. 260.

At the two endpoints $x = 0$ and $x = \pi$ of the range the function has to vanish anyway since these two points are common roots for all the functions of expansion. The essential roots of $\varphi_n(x)$ are therefore only

$$x_\alpha = \alpha \frac{\pi}{n} \quad (\alpha = 1, 2, \cdots n-1) \quad (3.9)$$

The roots are thus equidistant and the problem of trigonometric interpolation requires that we shall determine the coefficients $c_k$ of the approximation

$$\overline{f(x)} = c_1 \sin x + c_2 \sin 2x + \cdots c_{n-1} \sin (n-1)x \quad (3.10)$$

by the condition that in the $n - 1$ points (3.9) $\overline{f(x_k)}$ shall coincide with the given values $y_k = f(x_k)$ of the function.

The $n - 1$ conditions

$$\overline{f(x_k)} = f(x_k) \quad (3.11)$$

yield $n - 1$ linear equations by which the coefficients $c_k$ can be determined. Generally the solution of a simultaneous system of linear equations is a cumbersome process. But the equations (3.11) can be solved explicitly quite easily, due to the particular nature of the trigonometric functions. These functions are orthogonal not only with respect to the integrals

$$\int_0^\pi \varphi_i(x)\varphi_k(x)\,dx = 0 \qquad (i \neq k) \quad (3.12)$$

but also with respect to the sums:

$$\sum_{\alpha=1}^{n-1} \varphi_i(x_\alpha)\varphi_k(x_\alpha) = 0 \qquad \begin{pmatrix} i \neq k \\ i, k < n \end{pmatrix} \quad (3.13)$$

These conditions yield, in analogy to the formula (2.5) for the Fourier-coefficients, the following expression for $c_k$:

$$c_k = \frac{\sum f(x_\alpha)\varphi_k(x_\alpha)}{\sum \varphi_k^2(x_\alpha)} = \frac{2}{n} \sum_{\alpha=1}^{n-1} y_\alpha \sin\left(\frac{\pi k}{n}\alpha\right) \quad (3.14)$$

This formula is very well suited for numerical computation and permits to obtain the coefficients $c_k (k = 1, 2, \cdots n-1)$ without difficulty, even if $n$ runs as high as 24–30. The orthogonality of the trigonometric functions with respect to interpolation distinguishes them favorably from other orthogonal systems for which such a second orthogonality does not hold and, therefore, in order to solve the equations (3.11), the

general Lagrangean interpolation formula has to be employed, which usually involves more cumbersome calculations.

2. **Interpolation by cosine-functions.** We now consider the expansion of an *even* function $g(x)$ into a *cosine*-series. We write it in the form:

$$\overline{g(x)} = \tfrac{1}{2}c_0 + c_1 \cos x + c_2 \cos 2x + \cdots c_{n-1} \cos (n-1)x \qquad (3.15)$$

The orthogonal functions of the expansion are now

$$\varphi_k(x) = \cos kx \qquad (3.16)$$

with the zeros

$$x_\alpha = \frac{\pi}{2k}(2\alpha - 1) \qquad (\alpha = 1, 2, \cdots k) \qquad (3.17)$$

Hence the points of interpolation are now:

$$x_\alpha = \frac{\pi}{2n}(2\alpha - 1) \qquad (\alpha = 1, 2 \cdots n) \qquad (3.18)$$

The orthogonality condition (3.13) holds also for the cosine-functions, if the $x_\alpha$ are chosen in accordance with (3.18):

$$\sum_{\alpha=1}^{n} \varphi_i(x_\alpha)\varphi_k(x_\alpha) = 0 \qquad \begin{pmatrix} i \neq k \\ i, k < n \end{pmatrix} \qquad (3.19)$$

Hence we get for the coefficients of the expansion again:

$$c_k = \frac{\sum g(x_\alpha)\varphi_k(x_\alpha)}{\sum \varphi_k^2(x_\alpha)} = \frac{2}{n} \sum y_\alpha \cos \frac{\pi k}{2n}(2\alpha - 1) \qquad (3.20)$$

In the Fourier-analysis of empirical functions it would not be practical to employ different points of interpolation for the sine- and for the cosine-series. We prefer to read off the curve at the $2n + 1$ equidistant points:

$$x_\alpha = \alpha \frac{\pi}{n} \qquad (\alpha = -n, \cdots 0, \cdots + n) \qquad (3.21)$$

and then form separately the ordinates of the sine-series:

$$y_k = \tfrac{1}{2}[f(x_k) - f(-x_k)] \qquad (k = 1, 2, \cdots n-1) \qquad (3.22)$$

and the ordinates of the cosine-series:

$$y_k = \tfrac{1}{2}[f(x_k) + f(-x_k)] \qquad (k = 0, 1, 2, \cdots n) \qquad (3.23)$$

Now, if we use the points

$$x_\alpha = \frac{\pi}{n}\alpha \qquad (\alpha = 0, 1, 2, \cdots n) \tag{3.24}$$

as points of interpolation for the cosine-series of an even function $g(x)$, the orthogonality condition (3.19) will still hold for the new points $x_\alpha$ provided, however, that the two limiting ordinates $g(0)$ and $g(\pi)$ are introduced with *half weight*. We therefore define the quantities $y_k$ from very beginning as follows:

$$\begin{aligned} y_0 &= \tfrac{1}{2}g(0) \\ y_\alpha &= \tfrac{1}{2}g(x_\alpha) \\ y_k &= g(x_k) \qquad\qquad (k = 1, 2, \cdots n-1) \end{aligned} \tag{3.25}$$

Then again the formula holds, in complete analogy to (3.14) and (3.20):

$$c_k = \frac{2}{n}\sum_{\alpha=0}^{n} y_\alpha \cos\frac{\pi k}{n}\alpha \tag{3.26}$$

with the only difference that, having $n+1$ points of interpolation, we also need $n+1$ terms of the interpolating series, so that the series for $\overline{g(x)}$ has to be set up in the following form:

$$\overline{g(x)} = \tfrac{1}{2}c_0 + c_1\cos x + \cdots + \tfrac{1}{2}c_n\cos nx \tag{3.27}$$

3. **Relation between interpolating series and Fourier-series.** Another great advantage of the trigonometric interpolation is the fact that *an explicit relation can be established between the coefficients obtained by interpolation and the Fourier-coefficients of the infinite series.*

We want to distinguish the Fourier-coefficients from the coefficients of interpolation by denoting the former coefficients by $C_k$, the latter one by $c_k$. We again separate sine- and cosine-series and start with the sine-series.

Let us expand the odd function $f(x)$ in the range (3.5) into the infinite Fourier-series

$$\begin{aligned} f(x) &= C_1\sin x + C_2\sin 2x + \cdots \\ &= \sum_{\alpha=1}^{\infty} C_\alpha \sin\alpha x = \sum_{\alpha=1}^{\infty} C_\alpha\varphi_\alpha(x) \end{aligned} \tag{3.28}$$

The functions $\varphi_k(x)$ satisfy the following algebraic relation:

$$\tfrac{1}{2}[\varphi_{i+k}(x) + \varphi_{i-k}(x)] = \varphi_i(x) \cos kx \tag{3.29}$$

Owing to this relation we can substitute:

$$C_{n+k}\varphi_{n+k}(x) = -C_{n+k}\varphi_{n-k}(x) + 2C_{n+k}\varphi_n(x) \cos kx \tag{3.30}$$

With the help of such substitutions the series (3.28) may be rewritten in the following form:

$$f(x) = \sum_{k=1}^{n-1} c_k \varphi_k(x) + \varphi_n(x)F(x) \tag{3.31}$$

where we have put:

$$c_k = C_k - C_{2n-k} + C_{2n+k} - C_{4n-k} + C_{4n+k} - \cdots \tag{3.32}$$

Now the sum

$$\overline{f(x)} = \sum_{n=1}^{n-1} c_n \varphi_n(x) \tag{3.33}$$

represents an approximation of $f(x)$ which coincides with $f(x)$ at the zeros of $\varphi_n(x)$. This is exactly our series obtained by interpolating in the points (3.9). Hence the formula (3.32) represents *the explicit connection between the coefficients of interpolation* $c_k$ *and the Fourier-coefficients* $C_k$, deduced for the sine-series.

The corresponding formula for the *cosine*-series, using the *same* points of interpolation, is:

$$c_k = C_k + C_{2n-k} + C_{2n+k} + C_{4n-k} + C_{4n+k} + \cdots \tag{3.34}$$

while in the case of the $x_n$ defined by formula (3.18) we obtain

$$c_k = C_k - C_{2n-k} - C_{2n+k} + C_{4n-k} + C_{4n+k} - \cdots \tag{3.35}$$

4. **The number of approximating terms.** One remark seems to be necessary concerning the number $n$ of approximating terms. Owing to a widespread misconception, the number of ordinates needed for a close evaluation of the Fourier-coefficients by interpolation is frequently grossly over-estimated. In laboratories where Fourier-analysis is practised by interpolation, the notion prevails that the points $x_n$ in which we read off the given function $f(x)$ have to be sufficiently close to cover all essential details of the curve. One is particularly anxious that all the maxima and minima of the curve shall be carefully covered, which sometimes necessitates a prohibitive amount of work. The

reasoning behind this procedure is that the sums (3.14) and (3.26) are merely approximations of the integrals (2.5) which are needed in order to get the correct values of the Fourier-coefficients.[7a] Actually, these sums need not be considered as approximations of any integrals but as the *exact values of the coefficients of the interpolating series*. How close these coefficients come to the actual Fourier-coefficients, depends entirely on the more or less rapid *convergence* of the Fourier-series, and this is a point that can never be decided by mere inspection of the curve. If a Fourier-series has $n$ terms and no higher terms are present, then $n$ points will interpolate the function not only *approximately* but *exactly*. For example, the mere inspection of the curve

$$g(x) = \cos x - \tfrac{1}{2}\cos 2x + \tfrac{1}{3}\cos 3x \tag{3.36}$$

(between 0 and $\pi$) shows a rather rugged course and one would guess that at least 20 equidistant points are necessary to reproduce all essential details of the curve with some accuracy. And still but *four* points are actually needed (not indicating even roughly the course of the function) to get by interpolation *exactly* the four coefficients:

$$c_0 = 0, \qquad c_1 = 1, \qquad c_2 = -\tfrac{1}{2}, \qquad c_3 = \tfrac{1}{3} \tag{3.37}$$

Generally, of course, the given empirical curve will contain an infinite number of harmonic components. But ordinarily the higher harmonics decrease rapidly and the actual analysis will seldom include, within the limits of accuracy, a large number of overtones. Let us consider e.g. an acoustical analysis where empirical evidence has shown before that no overtones higher than 10th order are practically present. Then for the analysis of the sine-curve we need not more than 10, of the cosine-

[7a] Unfortunately the current text books on practical analysis seem to have adopted this erroneous concept. The excellent and most advisable book of *Runge-König*, Ref. 8, says on page 211 (in English translation): "If the function is not everywhere given but only in a finite number of points, we substitute the integrals by sums;" and on page 223: "Generally one will be able to tell by inspecting the curve of the function which shall be approximated, whether 12 ordinates are able to reproduce the course of the curve in all essential points. No considerable maxima and minima are allowed between two neighbouring ordinates." The very useful book of *Von Sanden*, Ref. 9, says on p. 128: "In the graphical and numerical methods of estimating the coefficient $a_n$ and $b_n$ about to be described, we assume that $f(x)$ is given at a series of discrete values through which a curve may be drawn." These remarks are not in accordance with the fact that the more or less rapid convergence of a Fourier-series cannot be judged with any reliability by mere inspection of the curve which represents the sum of the series.

curve 11 ordinates, so that it suffices if we divide the complete abscissa (betwen $-\pi$ and $+\pi$) into 20 equal parts. Naturally, one will prefer to leave some margin of safety and employ perhaps 12–14 ordinates for both curves, checking by computation that the coefficients $c_k$ above $k = 10$ are practically neglectable. But the further increase of the points in which we read off the function, is without any value. In fact, owing to the inevitable errors in reading off the ordinates, the results obtained by too many ordinates will be *less* reliable than the results obtained by the minimum number of ordinates.

In a given problem where we cannot guess a priori the number of practically active Fourier-coefficients, we will make a trial with a number $n$ which shall be chosen preferably too low than too high. Then we start with the evaluation of the *last* 2 or 3 coefficients $c_k$ and see whether they are above or below the limits of accuracy we require. If these $c_k$ are not yet neglectable, we have to increase the number of interpolating points, until we find that the highest Fourier-coefficients are already practically zero.

In the application of the trigonometric interpolation to the representation of analytical functions (see later) the number of interpolating points is even for high accuracy sometimes amazingly small. Take for example the series which replaces the Stirling series in representing the logarithm of the Gamma-function (see Chapter VI, par. 2). In the large range between $x = 1$ and $x = \infty$ only 10 basic values of the Gamma-function are needed (8 if we exclude the two end-points where the function is known without tabulation) in order to obtain an accuracy of $10^{-9}$ at any point of the range.[8]

## IV. Trigonometric interpolation and power series expansions

1. **Tshebysheff's polynomials.** A very excellent combination of the advantages of the Fourier-series and the power-series was introduced in approximation problems by *Tshebysheff*.[9] The function $f(x)$ shall be given in the range between

$$a = -1, \qquad b = 1 \tag{4.1}$$

[8] In the analysis of economic series we sometimes encounter curves of a very rugged nature, full of small maxima and minima which indicate a large number of high overtones. Such curves are not suited for the interpolation method and the mechanical analyzer gives better results than mere computation. But for smooth curves, particularly if high accuracy is required, the calculation competes successfully with the mechanical instruments.

[9] Concerning the literature on Tshebysheff's polynomials, cf. *Fréchet-Rosenthal*, Ref. 4, p. 1158.

We make the substitution

$$x = \cos t \tag{4.2}$$

and consider $f(x)$ as a function $F(t)$ of $t$ in the range between

$$0 \leqq t \leqq \pi \tag{4.3}$$

We can define $F(t)$ for the corresponding *negative* range as an *even* function

$$F(-t) = F(t) \tag{4.4}$$

without getting any discontinuities either of the function or of any of its derivatives at the point $t = 0$. Thus we can expand $F(t)$ into a pure cosine-series and stay within the natural range with respect to which the functions

$$\varphi_k(t) = \cos kt \quad (k = 0, 1, 2, \cdots) \tag{4.5}$$

form a complete orthogonal function-system. The very same functions, expressed in terms of the original variable $x$, are *polynomials* of the order $k$ of $x$, called "*Tshebysheff's polynomials.*" They are usually denoted by $T_k(x)$ so that we have the definition

$$T_k(x) = \cos (k \cos^{-1} x) \tag{4.6}$$

The range (4.1) is not always suitable for practical applications. We prefer at present to have some flexibility in the range of $x$ and leave the limits arbitrarily $a$ and $b$, instead of normalizing them to definite values. We thus define Tshebysheff's polynomials for our present purposes by the equation

$$T_k(x) = \cos kt \tag{4.7}$$

with the substitution

$$\cos t = \frac{2x - (b + a)}{b - a} \tag{4.7*}$$

Tshebysheff's polynomials permit to extend the outstanding properties of the Fourier-series to ordinary power series. *The Fourier-series of $F(t)$ in the variable $t$ is an ordinary power series of $f(x)$ in the variable $x$, arranged in Tshebysheff's polynomials $T_k(x)$.*

**2. Curve fitting by least squares.** In the problems of fitting a large number of observations by a smooth curve, the "method of least

squares" is customarily employed. Let us suppose that $f(x)$ can be represented by a power series of a given number of terms and the number of observations surpasses the number of coefficients at our disposal. Then, in order to get the "closest fit," we apply the principle that the square of the average error:

$$E^2 = \sum_{a=1}^{m} [f(x_a) - y_a]^2 \tag{4.8}$$

shall become as small as possible. Here $x_a$ are the points of observations and $y_a$ are the observed ordinates.

The minimum conditions for (4.8) yield a system of linear equations for the coefficients of the power series, (the "normal equations") which can be solved. The solution of these normal equations can be avoided and the coefficients of the expansion obtained explicitly, if we proceed as follows. We construct successively the "orthogonal polynomials" $P_k(x)$ of the order $k$ which have to satisfy the following conditions:

$$\sum_{a=1}^{m} P_i(x_a)P_k(x_a) = 0 \qquad (i \neq k) \tag{4.9}$$

We expand $f(x)$ in these polynomials:

$$f(x) = c_0P_0 + c_1P_1(x) + \cdots c_nP_n(x) \qquad (n \leq m - 1) \tag{4.10}$$

Then the coefficients of the "closest fit" are explicitly determined by the following equations:

$$c_k = \frac{\sum_{a=1}^{m} y_a P_k(x_a)}{\sum_{a=1}^{m} P_k^2(x_a)} \tag{4.11}$$

The construction of the orthogonal polynomials $P_k(x)$ requires considerable amount of work and they have to be computed for each value of $m$ (number of observations) separately.[10] But these cumbersome polynomials can be completely replaced by Tshebysheff's polynomials if we depart from the traditional custom of distributing the points $x_k$ of the observations in the ordinary *equidistant* manner. In statistical and actuarial work the equidistant spacing of the data is frequently unavoidable But in the laboratory measurements of

[10] Consult on this subject *Davis*, Ref. 2; *Jordan*, Ref. 5.

physics and engineering the customary equidistant spacing of the independent variable is generally not dictated by any inherent necessity. If we have a chance to dispose freely about the points $x_k$ at which the ordinates $y_k$ are read off, the problem of curve fitting in its mathematical aspects can be greatly simplified. *Tshebysheff's polynomials are automatically orthogonal with respect to the points*

$$x_k = \frac{a}{2}(1 - \cos t_k) + \frac{b}{2}(1 + \cos t_k) \tag{4.12}$$

where

$$t_k = \frac{\pi}{2m}(2k - 1) \qquad (k = 1, 2, \cdots n) \tag{4.12*}$$

[see the formula (3.19) together with the definition of Tshebysheff's polynomials owing to (4.7) and (4.7*)] *Hence the formula* (3.20) *of the trigonometric interpolation by a cosine-series automatically solves the problem of curve fitting in agreement with the "method of least squares" if the points of observation are chosen in an a priori given unequidistant fashion.*

For reasons of simpler computation a small modification is permissible in defining the values $x_k$. Let the formula (4.12) remain unchanged, but let the $t_k$ be chosen as follows:

$$t_k = \frac{\pi}{m}k \qquad (k = 0, 1, 2, \cdots n) \tag{4.13}$$

We again have $n$ unequal intervals but now the two end-points of the range:

$$x_0 = a, \qquad x_n = b \tag{4.14}$$

are included amongst the points of observation so that the number of observations is now $n + 1$. The polynomials of Tshebysheff still satisfy the condition of orthogonality, *provided that the weight factor* $\frac{1}{2}$ *is applied in counting the two limiting ordinates* $y_0$ *and* $y_n$. The formula (3.26) has now to be used for the evaluation of the coefficients $c_k$, and in using this formula we have to remember that the two limiting ordinates $y_0$ and $y_n$ are but *one-half* of the observed values.

For many problems of curve fitting and approximations it is practical to put the zero of the independent variable in the left end-point $a$ of the range. Moreover, the total breadth of the range can easily be normalized to 1 by choosing a proper unit in which the variable $x$ is

measured. Hence the following limits are particularly suited for practical applications:

$$a = 0, \qquad b = 1 \tag{4.15}$$

Considering the great usefulness of Tshebysheff's polynomials for approximation problems, we give in the following table the explicit expressions of the first 10 Tshebysheff's polynomials, adjusted to the range (4.15):

$$\begin{aligned}
T_0(x) &= 1 \\
T_1(x) &= 2x - 1 \\
T_2(x) &= 8x^2 - 8x + 1 \\
T_3(x) &= 32x^3 - 48x^2 + 18x - 1 \\
T_4(x) &= 128x^4 - 256x^3 + 160x^2 - 32x + 1 \\
T_5(x) &= 512x^5 - 1280x^4 + 1120x^3 - 400x^2 + 50x - 1 \\
T_6(x) &= 2048x^6 - 6144x^5 + 6912x^4 - 3584x^3 + 840x^2 - 72x + 1 \\
T_7(x) &= 8192x^7 - 28672x^6 + 39424x^5 - 26880x^4 + 9480x^3 - 1568x^2 \\
&\qquad + 98x - 1 \\
T_8(x) &= 32768x^8 - 131072x^7 + 212992x^6 - 180224x^5 + 84480x^4 \\
&\qquad - 21504x^3 + 2688x^2 - 128x + 1 \\
T_9(x) &= 131072x^9 - 589824x^8 + 1105920x^7 - 1118208x^6 + 658944x^5 \\
&\qquad - 228096x^4 + 44352x^3 - 4320x^2 + 162x - 1 \\
T_{10}(x) &= 524288x^{10} - 2621440x^9 + 5570560x^8 - 6553600x^7 + 4659200x^6 \\
&\qquad - 2050048x^5 + 549120x^4 - 84480x^3 + 6600x^2 - 200x + 1
\end{aligned} \tag{4.16}$$

This table can be constructed—and eventually continued—by recursion, making use of the simple *algebraic multiplication theorem* of the polynomials $T_k(x)$:

$$T_m(x)T_n(x) = \tfrac{1}{2}[T_{m+n}(x) + T_{m-n}(x)] \tag{4.17}$$

Putting $m = 1$ we get:

$$T_{n+1}(x) = 2T_n(x)T_1(x) - T_{n-1}(x) \tag{4.18}$$

and thus, starting with

$$T_0(x) = 1$$
$$T_1(x) = 2x - 1 \tag{4.19}$$

all the latter polynomials can be obtained successively.

Ordinarily the polynomials $T_k(x)$ are adapted to the range (4.1) which is, however, less suited for various approximation problems. One can show easily that the table (4.16) can be obtained by selecting all the polynomials of *even* order of the customary table, and then replace $x^2$ by $x$.

The following table gives the selected points $x_k$, adapted to the range (4.15):

$$x_k = \frac{1}{2}\left(1 + \cos\frac{\pi}{m}k\right), \tag{4.20}$$

to 4 decimal places, for all even values of $m$ between $m = 2$ and $m = 12$[11]):

| $m$ | |
|---|---|
| 2 | $x_0 = 1$ $x_1 = 0.5$<br>$x_2 = 0$ |
| 4 | $x_0 = 1, x_1 = .8536, x_2 = .5$<br>$x_3 = 0, x_4 = .1464$ |
| 6 | $x_0 = 1, x_1 = .9330, x_2 = .75, x_3 = .5$<br>$x_4 = 0, x_5 = .0670, x_6 = .25$ |
| 8 | $x_0 = 1, x_1 = .9619, x_2 = .8536, x_3 = .6913, x_4 = .5$<br>$x_5 = 0, x_7 = .0381, x_6 = .1464, x_5 = .3087$ |
| 10 | $x_0 = 1, x_1 = .9755, x_2 = .9045, x_3 = .7939, x_4 = .6545, x_5 = .5$<br>$x_{10} = 0, x_9 = .0245, x_8 = .0955, x_7 = .2061, x_6 = .3455$ |
| 12 | $x_0 = 1, x_1 = .9830, x_2 = .9330, x_3 = .8536, x_4 = .75, x_5 = .6294$<br>$x_{12} = 0, x_{11} = .0170, x_{10} = .0670, x_9 = .1464, x_8 = .25, x_7 = .3706$<br>$x_6 = .5$ |

(4.21)

[11] The symmetry of the trigonometric functions in all the four quadrants of the full circle grants particular advantages in calculating the coefficients $c_k$ of the trigonometric interpolation if $m$ is even. It is thus advisable to subdivide the range of the observations into an *even* number of intervals.

With the help of these tables the problem of curve fitting is now reduced to the following elementary procedure. We choose a suitable odd number $m + 1$ ($m$ being even) for the number of points at which the ordinates shall be recorded. Then we divide the given interval into $m$ unequal parts, according to the $x_k$ listed in the table (4.21). The ordinates $f(x_k)$ have to be observed in these selected $m + 1$ points. The recorded values are called $y_k$, with the exception of the two limiting ordinates belonging to $x_m = 0$ and $x_0 = 1$, where the recorded values have to be *halved* in order to get $y_m$ and $y_0$.

We now form the following series:

$$f(x) = \tfrac{1}{2} c_0 + c_1 T_1(x) + \cdots c_n T_n(x) \tag{4.22}$$

The order $n$ coincides with the order of the polynomial by which $f(x)$ shall be represented. That order is naturally smaller or at most equal $m$ (If $n$ happens to be equal $m$, the last coefficient $c_n$ shall be introduced with the factor 1/2, like in formula (3.27).) The coefficients $c_k$ are explicitly determined by the formula (3.26). After evaluating the coefficients $c_k$ we form the series (4.22) and finally rearrange the series to an ordinary power series according to the table (4.16), obtaining finally $f(x)$ in the form:

$$f(x) = a_0 + a_1 x + \cdots a_n x^n \tag{4.23}$$

We see that the convenient and for numerical work exceedingly agreeable formulae of the trigonometric interpolation are able to solve rather simply the ordinarily much more cumbersome problem of "curve fitting by least squares," avoiding completely the laborious solution of the normal equations that we ordinarily encounter in such problems.[12] The only condition is that we have to observe the curve at certain definitely chosen unequidistant points $x_k$ instead of the customary equidistant spacing. This condition can frequently be satisfied without greater inconveniences. But if we prefer equidistant spacing, we can choose our points close enough and then employ the method described in the next paragraph.

[12] In fact even the trigonometric interpolation need not be carried out to full extent, since $n$ is usually much smaller than $m$. All the coefficients higher than $c_n$ would come out automatically as zero if it would not be for experimental errors. The evaluation of the higher coefficients is useful for the estimation of the probable error of the lower coefficients, caused by experimental errors. Eventually a systematic departure of $f(x)$ from the assumed law may be detected if the coefficient $c_{n+1}$ turns out to be larger than it should be expected by the inaccuracy of the measurements.

3. **Curve fitting of smooth functions.** Considering the great accuracy of modern recording instruments the full application of the method of least squares is but seldom required. If the experimental errors are small and the observations sufficiently close, the procedure of the previous paragraph can be modified and the unequal spacing of the points of observation avoided. We record the ordinates $y_k$ in close *equidistant* steps and set up a difference table. Then a "smoothing process" can be applied to get rid of the small unevenness of the curve due to the experimental errors. This simple and effective process is described in *Von Sanden's* "Practical Analysis", Ref. 9, page 121. Owing to the close steps by which the abscissa proceeds, the *fourth* differences $\Delta^4 f(x)$ can be considered as merely the result of experimental errors, and this assumption may be checked by the irregular nature of the fourth column of the difference table. Then the correction

$$-\frac{3}{35}\Delta^4 f(x) \tag{4.24}$$

is applied on each observed ordinate $y_k$. With these corrected values a new difference table can be set up. The smoothing of the new curve will manifest itself by the smallness of the still remaining 4th differences.

With the help of the corrected difference table the observed function $f(x)$ may be interpolated at *any* point of the range, with the help of the Gregory-Newton interpolation formula (1.2), employing 4 terms. We select again our points $x_k$ of the trigonometric interpolation according to the table (4.21) but now we determine the values of the ordinates $y_k$ at the selected points $x_k$ *by calculation*, (interpolation), instead of direct observation. The remaining part of the procedure coincides with the procedure of the previous paragraph. The advantage of the present method is that the unequal spacing of the points of interpolation enters only in the calculation but the observations themselves are made in *equidistant* steps. The only conditions for the success of this simple method are: sufficiently small experimental errors and sufficiently smooth $f(x)$, i.e. sufficiently close steps.

4. **Comparison with Legendre's polynomials.** If an empirical function $f(x)$ is tabulated and can be interpolated practically at any point of a given range, the current "method of least squares" suggests the application of Legendre's polynomials for the problem of approximating that function by powers. We normalize the range to

$$a = -1, \quad b = 1 \tag{4.24}$$

and set up the series

$$\overline{f(x)} = \gamma_0 + \gamma_1 P_1(x) + \cdots \gamma_n P_n(x) \tag{4.25}$$

where $P_k(x)$ are Legendre's polynomials. The coefficients $\gamma_k$ have then to be determined by the formula

$$\gamma_k = (k + \tfrac{1}{2}) \int_{-1}^{+1} f(x) P_k(x)\, dx \tag{4.26}$$

Since $f(x)$ is not given in analytical form, the integrals (4.26) are usually obtained by numerical integration, i.e. by substituting the integrals by sums. For this process many more terms have to be engaged than for the sums (3.20). The latter sums are not merely the approximations of integrals but the exact values for the coefficients of a series which is obtained by interpolation. Now the interpolation method may also be adopted for Legendre's polynomials, interpolating the series (4.25) at the zeros of the first neglected polynomial, i.e. at the zeros of $P_{n+1}(x)$. This cuts down the laborious task of numerical integrations and involves a much smaller number of ordinates $y_k$. But still the Lagrangean interpolation formula has to be applied for the evaluation of the coefficients $a_k$ of the resulting power series (4.23). This task is still much more laborious than the trigonometric interpolation. The formulae of the trigonometric interpolation (3.20) have no analogy in working with other orthogonal polynomials because these formulae are based, as we have seen above, on an independent "second orthogonality property" of the Fourier functions which has no analogy for other orthogonal functions. This feature distinguishes the trigonometric interpolation favorably and renders it better suited for practical applications than any other interpolation.

In addition to the simplified method of calculation there is a second reason which speaks for the superiority of Tshebysheff's polynomials in comparison with Legendre's polynomials. Let us suppose that the same function $f(x)$ is expanded once in Tschebysheff and once in Legendre polynomials. We then keep in both cases the first $n + 1$ terms of the expansion and neglect the rest. We assume that both series are sufficiently convergent so that the simple estimation (2.11) of the error can be applied. Then the error of the Legendre series is

$$\overline{\eta^P(x)} = \gamma_{n+1} P_{n+1}(x) \tag{4.27}$$

and the error of the Tshebysheff series:

$$\overline{\eta^T(x)} = C_{n+1} T_{n+1}(x) \tag{4.28}$$

Both errors have the same oscillatory character. They differ, however, in the behaviour of the *amplitude*. The functions $P_k(x)$ have a variable amplitude which assumes the maximal value 1 at both ends of the range and decreases toward the middle of the range. The amplitude of the fluctuations of Tshebysheff's polynomials, on the other hand, *is constantly equal* 1. Thus the error has the tendency to *increase* toward the end of range in the case of Legendre's polynomials while it has the tendency to be distributed *uniformly* over the range in the case of Tshebysheff's polynomials. It is true that Legendre's polynomials guarantee a smaller *average* error. *But the greatest possible error at any point of the range is reduced in using Tshebysheff's polynomials*, owing to the more uniform distribution of the errors over the complete range, which eliminates the increase of the error toward the end of the range. In fact, Tshebysheff discovered his polynomials in connection with the problem, how to find an approximation which reduces the maximal possible error at any point of the range to a minimum.[13]

### V. Economization of a given power series

If an empirical function has been interpolated by the method of the previous paragraph, the inspection of the coefficients $c_k$ of the Tshebysheff-series will show it directly, how many terms of the series have to be kept for a certain required accuracy, no matter whether the accuracy is determined by the accuracy of the measurements or by other view points. Those terms of the series, the coefficients of which are below the required accuracy, even if the sum of their absolute values is taken in account, can be neglected. If a power series is arranged in Tshebysheff's polynomials, then it is reduced already to its *most economical form*. But frequently a power series is given whose economy can be greatly improved because one is able to replace the given series by another series of a *smaller number of terms*, without sacrificing essentially in accuracy.

The range of our problem shall again be limited to the convenient range 0 to 1, in accordance with (4.15). The given power series shall be

$$y = a_0 + a_1 x + \cdots a_n x^n \tag{5.1}$$

The last coefficient $a_n$ shall not be below the limit of accuracy and thus the term $a_n x^n$ is not neglectable. And still the reduction of the series to a polynomial of lower order may be possible.

[13] See Ref. 1, p. 75; Ref. 4, p. 1158.

Considering the fact that—according to the table (4.16)—the highest coefficient of Tshebysheff's polynomials $T_k(x)$ is $2^{2k-1}$ we may write

$$\frac{T_n(x)}{2^{2n-1}} = x^n + \alpha_{n-1}x^{n-1} + \cdots \alpha_0 \tag{5.2}$$

Thus

$$x^n = -(\alpha_{n-1}x^{n-1} + \cdots \alpha_0) + \frac{T_n(x)}{2^{2n-1}} \tag{5.3}$$

This equation shows that, instead of neglecting $x^n$, we may reduce $x^n$ to lower powers, with an amazingly great efficiency. If we put

$$x^n = -(\alpha_{n-1}x^{n-1} + \cdots \alpha_0), \tag{5.4}$$

we have committed an error which does not exceed at any point of the range the amount of $2 \cdot 4^{-n}$. Thus, although $a_n$ itself may be far *above* the limit where it can be neglected, the multiplication by $2 \cdot 4^{-n}$ may diminish it to far *below* the limit. Hence, replacing $x^n$ by (5.4), the original series (4.20) may be reduced to a new series of $n - 1^{st}$ order. Now the process may be repeated again for $x^{n-1}$ and eventually the last coefficient, after multiplying by $2 \cdot 4^{-(n-1)}$, may still drop below the required limit. This process of successive reductions can be continued until the last coefficient $a_m$ of the series is already so large that even the multiplication by $2 \cdot 4^{-m}$ does not diminish it sufficiently. *Then the reduction process is completed and the series is reduced—for the given accuracy—to its most economical form.* Since we know exactly the error committed at each reduction, we can estimate the resulting error by the absolute sum of all the single errors, committed at each step.

Instead of these successive reductions, the following systematic method yields the same result and is particularly advisable if the number of terms which can be saved is large. *We express the powers of $x$ in Tshebysheff's polynomials and rearrange the given series in $T_k(x)$.*

$$y' = \tfrac{1}{2}c_0 + c_1T_1(x) + \cdots c_nT_n(x) \tag{5.5}$$

The $c_k$ coefficients decrease much faster than the original coefficients $a_k$ did. From a certain $c_m$ on we neglect all the higher terms if even the sum of the absolute values of all the neglected coefficients stays under the required limit. Then we replace in the remaining series the $T_k(x)$ again by the corresponding polynomials, making use of the table (4.16), and rearrange the series finally to an ordinary power-series. The new series has a smaller number of coefficients than the original series had, and the coefficients are completely different.

The following table expands the powers of $x$ in Tshebysheff's polynomials, until $x^{10}$:

$$1 = T_0$$

$$x = \tfrac{1}{2}(T_0 + T_1)$$

$$x^2 = \tfrac{1}{2}(3T_0 + 4T_1 + T_2) \qquad (5.6)$$

$$x^3 = \tfrac{1}{2^2}(10T_0 + 15T_1 + 6T_2 + T_3)$$

$$x^4 = \tfrac{1}{2^3}(35T_0 + 56T_1 + 28T_2 + 8T_3 + T_4)$$

$$x^5 = \tfrac{1}{2^4}(126T_0 + 210T_1 + 120T_2 + 45T_3 + 10T_4 + T_5)$$

$$x^6 = \tfrac{1}{2^5}(462T_0 + 792T_1 + 495T_2 + 220T_3 + 66T_4 + 12T_5 + T_6)$$

$$x^7 = \tfrac{1}{2^6}(1716T_0 + 3003T_1 + 2002T_2 + 1001T_3 + 364T_4 + 91T_5 + 14T_6 + T_7)$$

$$x^8 = \tfrac{1}{2^7}(6435T_0 + 11440T_1 + 8008T_2 + 4368T_3 + 1820T_4 + 560T_5 + 120T_6 + 16T_7 + T_8)$$

$$x^9 = \tfrac{1}{2^8}(24310T_0 + 43758T_1 + 31824T_2 + 18564T_3 + 8568T_4 + 3060T_5 + 816T_6 + 153T_7 + 18T_8 + T_9)$$

$$x^{10} = \tfrac{1}{2^9}(92378T_0 + 167960T_1 + 125970T_2 + 77520T_3 + 38760T_4 + 15504T_5 + 4845T_6 + 1140T_7 + 190T_8 + 20T_9 + T_{10})$$

The mathematical law of these coefficients can easily be stated. The common divisor of the expansion of $x^k$ is $\frac{1}{2} \cdot 4^k$.[14] The factor of $T_j$ inside of the parenthesis of the same expansion is the binomial coefficient

$$\binom{2k}{k-j} = \frac{(2k)!}{(k-j)!\,(k+j)!} \qquad (j \leqq k) \qquad (5.7)$$

with the only exception of $T_0$ the coefficient of which is but *one half* of the general expression.

The following example illustrates the application of the rearrangement method.

*Problem*: In the range between $x = 0$ and $x = 2\pi$ a power series is required which shall represent $y = \sin x$ with an accuracy of $\pm .01$ with the smallest number of terms.

[14] We conveniently start the process of rearrangement of the series (5.1) by dividing at first all the $a_k$ by $\frac{1}{2} \cdot 4^k$. We then have only multiplications by integers and no divisions any more.

At first we normalize the range to the standard limits by changing the scale of $x$. We call the original variable $\xi$ and put

$$\xi = 2\pi x \tag{5.8}$$

Then we have

$$f(x) = \sin 2\pi x \tag{5.9}$$

In the Taylor-expansion of sin $2\pi x$ we have to go as high as $x^{19}$ until the error becomes smaller than 0.01. Thus we start with the following "uneconomical" series:

$$\begin{aligned} \overline{f(x)} = 2\pi x - \frac{(2\pi)^3}{3!}x^3 + \frac{(2\pi)^5}{5!}x^5 - \cdots \\ + \frac{(2\pi)^{17}}{17!}x^{17} - \frac{(2\pi)^{19}}{19!}x^{19} \end{aligned} \tag{5.10}$$

The rearrangement of the series in Tshebysheff's polynomials yields:[15]

$$\begin{aligned} \overline{f(x)} = &- 0.0001T_0 - 0.5695T_1 - 0.0002T_2 + 0.6667T_3 \\ &- 0.0001T_4 - 0.1044T_5 - 0.0000T_6 + 0.0068T_7 + \cdots \end{aligned} \tag{5.11}$$

Thus with a maximal error of $\pm 0.007$ we can put

$$\overline{f(x)} = -0.569T_1 + 0.667T_3 - 0.104T_5 \tag{5.12}$$

Expressing $T_k(x)$ again in powers of $x$ we obtain finally the following approximation:

$$\begin{aligned} \sin 2\pi x = 0.007 + 5.642x + 9.758x^2 - 95.594x^3 + 133.632x^4 \\ - 53.453x^5 \quad \pm(0.008) \end{aligned} \tag{5.13}$$

The original polynomial of 19th order has been reduced, within the limits of accuracy, to a polynomial of only 5th order.

This method of rearranging a given power series in Tshebysheff's polynomials and then, after dropping the neglectable terms, arrange back again in powers of $x$ is practically only advisable if the number of terms in the given series is neither too small nor too large. For a short power series the before-mentioned *successive* reduction process is frequently preferable since it avoids the changing from $x^k$ to $T_k(x)$ and then back to $x^k$ again. On the other hand, if the number of terms gets too large, we better make use of the same method of "trigonometric

[15] The very small coefficients of all the even $T_k(x)$ find their explanation in the fact that in the *infinite* expansion of sin $2\pi x$ in the $T_k(x)$ all the even $C_k$ drop out.

interpolation" we have employed in the approximation of empirical functions. We then use the series only in obtaining the $y_k$-values at the characteristic points $x_k$ of the trigonometric interpolation (see table (4.21)). With these $y_k$ we go again in the formulae (3.26) and obtain directly the $c_k$ of the rearranged series. This method recommends itself by the relatively small number of points in which the function has to be known and by the simple structure of the formulae (3.26) which are better adapted for numerical computation than the rearrangement of a long series.

### VI. Trigonometric interpolation of analytical functions

The method of trigonometric interpolation is customarily limited to the representation of empirical functions. The Fourier-series is particularly adapted to the representation of non-analytical functions or functions whose analytical formulation is not available. The purpose of the present investigation is to show how successfully the same methods apply to the representation of purely *analytical* functions. The classical method of representing analytical functions is the expansion in a power series. These expansions have an *extrapolatory* character. On the other hand, Tshebysheff's polynomials bring power series and Fourier series in close contact with each other and permit to apply the *interpolatory* methods of the Fourier series to analytical functions. This results in a new type of power series expansions which is more efficient than the ordinary extrapolating series by which analytic functions are ordinarily represented. By the application of these principles the convergence of the Taylor series can be greatly improved. In the case of another type of expansions, called "asymptotic expansions," the gain is twofold: not only is the convergence improved if we compare the same number of terms of the old and the new series, but the error *converges monotonically toward zero* with increasing number of terms, whereas the asymptotic series reduces the error at first with increasing number of terms, but then a finite limit is attained and the error begins to increase again.

For the sake of a short manner of speaking, we want to use the term "economization" in the sense that a power series, which approximates a given function, may be replaced by another power series which yields, within a certain definite range, a smaller error for the same number of terms. In the present chapter we confine ourselves to a definite *real* range of the variable $x$. The extension of the power series to the complex domain shall be the subject of chapter VII.

1. **Economization of the Taylor-series.** We consider at first the Taylor-series and distinguish between two possibilities: the range in which the given analytical function $f(x)$ shall be represented may be

a) inside

b) partly outside

of the range in which the Taylor-series converges.

For the present theoretical discussion we want to put the origin of our reference system in the middle of the given range and normalize the limits to (4.1). In order to combine Fourier-series and power-series we introduce the angle variable $t$ by putting

$$x = \cos t \tag{6.1}$$

The function $f(x)$ between the limits $x = -1$ to $x = 1$ is now a function $F(t)$ between the limits $t = 0$ to $t = \pi$. We expand this function with respect to the complete orthogonal function system $\varphi_k = \cos kt$. That means in the original variable $x$ the infinite series:

$$f(x) = \tfrac{1}{2}C_0 + C_1 T_1(x) + C_2 T_2(x) + \cdots \tag{6.2}$$

The absolute convergence of this series is guaranteed by the regularity of $f(x)$.

If the range (4.1) is inside of the range of convergence of the Taylor-expansion about the point $x = 0$, the coefficients $C_k$ may be obtained by mere rearrangement of the Taylor-series

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots \tag{6.3}$$

expressing $x^k$ in terms of Tshebysheff's polynomials. This method was described in Chapter V in connection with a *finite* series but it holds without alteration for an infinite series, too, provided the given series is convergent all over the range (4.1). The advantage of the series (6.2) compared with the series (6.3) is its *quicker convergence*, that means: for a given accuracy a smaller number of terms is sufficient. Neglecting all the terms above a certain order $n$ in the series (6.2) we can rearrange the series again in powers of $x$ and thus obtain, instead of the Taylor series, a new series with modified coefficients which approximates closer than the original series, if we take in both cases the same number of terms. To be sure, *the coefficients of the new series are all the time changing as n changes* while the Taylor-series has rigid

coefficients. The series arranged in $T_k(x)$ has also rigid coefficients but not the series that we obtain by rearranging the first $n + 1$ terms of the infinite series (6.2) in powers of $x$. The greater efficiency of these series may be explained by the fact that the coefficients of the successive approximations are not a priori established rigid quantities like the coefficients of the Taylor-series but are permanently changing with the number of approximating terms. They are adapted to the range in which, and to the number of powers by which, we approximate.

If the radius of convergence of the Taylor-series is smaller than 1 because there is a singularity of $f(z)$ inside of the unit-circle—although $f(x)$ is still everywhere regular along the real axis—we are encountered with the case *b*) where the chosen range of approximation is partly *outside* of the realm of convergence of the Taylor-series. The series (6.2) which is equivalent to the Fourier-expansion of $F(t)$, between 0 and $\pi$, is still everywhere absolutely convergent, owing to the fundamental properties of orthogonal expansions. But now we can no longer obtain the coefficients of the series (6.2) by merely rearranging the Taylor-series (6.3). The nature of the successive approximations, however, is again the same as before: taking the first $n + 1$ terms of the series (6.2) and rearranging them in powers of $x$ we get a sequence of series which approximate the function $f(x)$ everywhere inside of the chosen range $-1$ to $+1$ (including the limits) arbitrarily closely, although the Taylor-series fails completely in one portion of the range. The *interpolating* nature of the new series and the *flexibility of the coefficients* is again responsible for the greater economy of the series.

In order to get the coefficients $C_k$ in the present case we should make use of the classical method of determining the coefficients of a Fourier-expansion. The classical formula, written in the variable $x$, gives:

$$C_k = \frac{2}{\pi} \int_{-1}^{+1} f(x) T_k(x) \frac{dx}{\sqrt{1 - x^2}} \tag{6.4}$$

Actually these integrals are but seldom practically available. Even relatively simple analytic functions lead to very complicate forms of definite integrals which cannot be easily handled. But, fortunately, other devices are available which yield the $C_k$ indirectly, if not exactly but with sufficient closeness. The method of trigonometric interpolation is particularly advisable, due to the small number of terms which enter in the summation process that replaces the integration of the

formula (6.4). If the series (6.2) has a rapid convergence, the function $f(x)$ has only to be given in a small number of fundamental points

$$x_k = \cos \frac{\pi}{n} k \qquad (k = 0, 1, 2, \cdots n) \tag{6.5}$$

in order to evaluate the coefficients $c_k$ of the approximating series

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots + \tfrac{1}{2}c_n T_n(x) \tag{6.6}$$

The method in this form coincides exactly with the method discussed above in approximating empirically given functions, (see IV, 2 and 3) the only difference being that the "ordinates" $y_k = f(x_k)$ are now the *calculated* values of $f(x)$ at the selected points $x_k$, and not the results of measurements.

This purely numerical method can frequently be avoided and substituted by more analytical means. In many instances the coefficients $c_k$ of the approximating series (which are not identical with the true Fourier coefficients $C_k$ but only approach them) can be obtained, without making any explicit use of the actual numerical values of $f(x)$, merely by solving a system of *recursion formulae* similar to those by which the coefficients of the Taylor-series can be determined. This is particularly true if $f(x)$ is defined as the solution of a linear differential equation with rational coefficients. The methods then available will be discussed in chapter VIII. At present we consider a purely *algebraic* example which does not involve any differentiations. The example is exceedingly suited for the demonstration of the different behaviour of extrapolating and interpolating infinite series. At the same time the obtained expansion is quite fundamental for the problem of the next chapter: the extension of a series, obtained by trigonometric interpolation, to the complex domain.

*Problem*: The function

$$f(x) = \frac{1}{a - x} \tag{6.7}$$

shall be expanded in Tshebysheff's polynomials. The constant $a$ may be any real or complex number which is not on the portion $-1$ to $+1$ of the real axis.

Before obtaining the infinite expansion of $f(x)$ we discuss the *interpolation* of $f(x)$, choosing the points of interpolation as the zeros of $T_n(x)$. We set up the following equation:

$$\frac{1}{a - x} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots + c_{n-1} T_{n-1}(x) + \tau \frac{T_n(x)}{a - x} \tag{6.8}$$

The possibility of this equation is established by the fact that the quotient

$$\frac{T_n(a) - T_n(x)}{a - x} \tag{6.9}$$

is a polynomial of $x$ of $n - 1$st order. This determines $\tau$ directly to

$$\tau = \frac{1}{T_n(a)} \tag{6.10}$$

The expansion at the right of (6.8), leaving out the last term, obviously coincides with the approximation

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots c_{n-1} T_{n-1}(x) \tag{6.11}$$

obtained by interpolating $f(x)$ at the zeros of $T_n(x)$. Indeed: if $T_n(x)$ vanishes, $f(x)$ and $\overline{f(x)}$ coincide.

The possibility of determining the $c_k$ by mere recursion is based on the simple *algebraic multiplication theorem* (4.17) of Tshebysheff's polynomials. We multiply the equation (6.8) by $a - x$, replacing $x$ by $T_1(x)$.[16] The multiplication theorem (4.17) combined with collecting terms yields the following recursion formulae for the coefficients $c_k$ :

$$ac_k - \tfrac{1}{2}(c_{k+1} + c_{k-1}) = 0 \qquad (k = 1, 2, \cdots n - 1) \tag{6.12}$$

$$\tfrac{1}{2}ac_0 - \tfrac{1}{2}c_1 = 1 \tag{6.13}$$

The first set of equations may be solved by the assumption:

$$c_k = Aq^k \tag{6.14}$$

which yields for $q$ the quadratic equation

$$q^2 - 2aq + 1 = 0 \tag{6.15}$$

with the roots

$$q_1 = a + \sqrt{a^2 - 1}, \qquad q_2 = a - \sqrt{a^2 - 1} \tag{6.16}$$

Thus

$$c_k = A_1 q_1^k + A_2 q_2^k \tag{6.17}$$

But $c_n$ must be zero. This yields the following relation between the two constants $A_1$ and $A_2$ :

$$A_1 q_1^n + A_2 q_2^n = 0 \tag{6.18}$$

[16] Our range is at present the classical range $-1$ to $+1$ and thus the traditional definition (4.6) holds. The polynomials listed in table (4.16) are adjusted to the range 0 to 1 and are thus not identical with the present $T_k(x)$.

and we may put

$$c_k = B(q_1^n q_2^k - q_2^n q_1^k) \tag{6.19}$$

or considering that

$$q_1 q_2 = 1 \tag{6.20}$$

we have

$$c_k = B(q_1^{n-k} - q_2^{n-k}) \tag{6.21}$$

Then the condition (6.13) determines $B$ and we obtain

$$B = \frac{2}{\sqrt{a^2 - 1}} \frac{1}{q_1^n + q_2^n} \tag{6.22}$$

which yields finally:

$$c_k = \frac{2}{\sqrt{a^2 - 1}} \frac{q_1^{n-k} - q_2^{n-k}}{q_1^n + q_2^n} \tag{6.23}$$

An interesting modification of this formula is possible which will be of importance for the later application of our series in chapter VII. Let us introduce the complex angle $\theta$ by putting

$$a = \cos \theta \tag{6.24}$$

then

$$q_1 = e^{i\theta}, \qquad q_2 = e^{-i\theta} \tag{6.25}$$

$$B = \frac{1}{i \sin \theta} \frac{1}{\cos n\theta} \tag{6.26}$$

$$c_k = \frac{2 \sin (n-k)\theta}{\sin \theta \cos n\theta} \tag{6.27}$$

Considering the definition of Tshebysheff's polynomials due to (4.6) we can put

$$\cos n\theta = T_n(a) \tag{6.28}$$

$$\frac{\sin (n-k)\theta}{\sin \theta} = \frac{1}{n-k} T'_{n-k}(a) \tag{6.29}$$

We thus obtain the following expansion:

$$\frac{1}{a-x} = \frac{T'_n(a)}{nT_n(a)} + \frac{2}{T_n(a)} \sum_{\alpha=1}^{n-1} \frac{T'_{n-\alpha}(a)}{n-\alpha} T_\alpha(x) \tag{6.30}$$

with the remainder:

$$\frac{T_n(x)}{T_n(a)}\frac{1}{a-x} \tag{6.31}$$

The result may be written more symmetrically in form of the following formula:

$$\frac{T_n(a)-T_n(x)}{a-x}=\frac{T'_n(a)}{n}+2\sum_{\alpha=1}^{n-1}\frac{T'_{n-\alpha}(a)}{n-\alpha}T_\alpha(x) \tag{6.32}$$

This formula could have been derived also with the help of Lagrange's interpolation formula or by the summation of a geometrical series if instead of $x$ the angle variable $t$ would have been introduced. But we wanted to follow a procedure which is applicable to a large class of problems: *to obtain the coefficients of the interpolating series by the solution of recursion formulae.*

What happens with the series (6.30) if $n$ increases toward infinity? We consider the original expression (6.23). One of the two roots $q_1$, $q_2$ will have the larger absolute value. Let us suppose it is $q_1$. Then with increasing $n$ the powers of $q_2$ will be practically neglectable compared with $q_1$ and, as $n$ approaches infinity, we get in the limit:

$$C_k=\frac{2}{\sqrt{a^2-1}}\frac{1}{q_1^k}=\frac{2}{\sqrt{a^2-1}}q_2^k \tag{6.33}$$

and the following infinite expansion results:

$$\frac{1}{a-x}=\frac{1}{\sqrt{a^2-1}}\left(1+2\sum_{\alpha=1}^{\infty}q_2^\alpha T_\alpha(x)\right) \tag{6.34}$$

The alternative formula, if $q_2$ is absolutely larger, follows:

$$\frac{1}{a-x}=\frac{-1}{\sqrt{a^2-1}}\left(1+2\sum_{\alpha=1}^{\infty}q_1^\alpha T_\alpha(x)\right) \tag{6.34*}$$

We finally want to introduce again the modified range (4.15) for the power expansion. For this purpose we replace $x$ by $2x-1$. The new $x$ now varies between 0 and 1 and the old $T_k(2x-1)$ become identical with the $T_k(x)$ listed in table (4.16). Now the following formula results:

$$\frac{1}{a-x}=\frac{\pm 1}{\sqrt{a^2-a}}\left(1+2\sum_{\alpha=1}^{\infty}q^\alpha T_\alpha(x)\right) \tag{6.35}$$

where

$$q = 2a - 1 \mp 2\sqrt{a^2 - a} \tag{6.36}$$

and the upper or lower sign is decided by the condition that the root with the smaller absolute value has to be chosen.

As a special application of the formula (6.35) we put

$$a = -1 \tag{6.37}$$

and obtain the following interesting expansion:

$$\frac{1}{1+x} = \sqrt{2}(\tfrac{1}{2} - pT_1(x) + p^2T_2(x) - \cdots) \tag{6.38}$$

where

$$p = 3 - 2\sqrt{2} = 0.17157\cdots \tag{6.39}$$

The ratio of two successive coefficients of the series is constant, similarly to the ordinary Taylor-development:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots \tag{6.40}$$

But here that ratio is $-1$ whereas in the development before it is only about $\frac{1}{6}$ of that value. Both series hold in the same realm

$$0 \leqq x \leqq 1 \tag{6.41}$$

An interesting discussion can be linked up with the two series (6.38) and (6.40), illustrating the divergent nature of interpolating and extrapolating series. Let us keep the first seven terms of the series (6.38) and drop the rest. We substitute for $T_s(x)$ their expressions according to the table (4.16) and obtain the following approximation of the function

$$f(x) = \frac{1}{1+x}: \tag{6.42}$$

$$\overline{f(x)} = 0.9999925 - 0.9992202x + 0.9863809x^2 - 0.9070856x^3 + 0.6753632x^4 - 0.3293104x^5 + 0.0738853x^6 \tag{6.43}$$

The error within the given range exceeds nowhere the amount of $6.10^{-6}$.

We now integrate and obtain the following approximation of the function

$$f(x) = \log(1 + x): \tag{6.44}$$

$$\overline{f(x)} = 0.9999925x - 0.4996101x^2 + 0.3287936x^3 - 0.2267714x^4 + 0.1350726x^5 - 0.0548851x^6 + 0.0105550x^7 \quad (6.45)$$

The error at any point of the range is limited by

$$\epsilon < \pm 10^{-6} \quad (6.46)$$

The Taylor-expansion of the same function is

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \quad (6.47)$$

Here the error is smaller than the first neglected term. In order to guarantee the exactness of the series (6.45) for the complete range (6.41), 1 million terms of the Taylor-series would be required.

This example shows quite characteristically the great superiority of an interpolating series for the purposes of an approximation in the large. The Taylor-series (6.47) produces the function $f(x)$ with the help of the value of the function and all its derivatives *at the point* $x = 0$. The values of $f(x)$ in the distance are determined by knowing $f(x)$ only *at a certain point and its infinitesimal surroundings*. This is *extrapolation*. The convergence of the series slows down more and more as we move away from the center of expansion and is tremendously retarded if we approach the boundary of the circle of convergence.[17]

Let us consider, on the other hand, the series (6.38). This series has an *interpolating* nature as one can readily see by estimating the error of the sum of the first $n$ terms of the series. The points of interpolation at which function and approximation coincide are very near to the zeros of the first neglected polynomial, i.e. $T_n(x)$. All these zeros are real and within the given range, although they are not evenly distributed over the range. But their number is $n$ and thus one can define uniquely an approximation by a polynomial of $n - 1^{st}$ order by the condition that function and approximation shall coincide at the zeros of $T_n(x)$. The formula (6.11) with the coefficients (6.23) solves this interpolation problem. These coefficients $c_k$ are slightly different

[17] The "fairness" of comparison may be objected to by the fact that the point of expansion of the Taylor-series has been chosen at the end-point $x = 0$ of the range, instead at the middle $x = \frac{1}{2}$, which would have lessened the distance of extrapolation and thus weakened the favorable balance of the interpolating series. The author believes, however, that the example in the form above is actually characteristic, because in so many instances the Taylor-expansion is actually only available at the one *end-point* of the range while the expansion about the middle of the range has coefficients which are practically not accessible.

from the coefficients $C_k$ of the infinite series (6.34) but the closeness of approximation is in both cases essentially the same. The infinite series (6.34) has still the nature of an interpolating series, with slightly shifted positions of the points of interpolation.

Of noticeable interest is also the fact that the seven terms of the series (6.45) and the million terms of the series (6.47) approximate both the *same* function $\log(1+x)$ so that the difference between these two series of widely different coefficients and number of terms is less than $10^{-6}$ at any point $x$ between 0 and 1. These two power series, the one of a small, the other of a large number of terms, represent two very different functions *outside* of the chosen range but are practically identical for a whole finite interval. The difference $\epsilon$ which was here smaller than $10^{-6}$ in its absolute value, may decrease to any arbitrarily small amount. And still we will have two series of a different number of terms and entirely different coefficients. Their sum approach each other indefinitely but their coefficients never coincide.

This behaviour seems to contradict a well known theorem proved in the theory of analytical functions of a complex variable.[18] According to this theorem there is only *one* infinite power series into which a given analytical function can be expanded. If we have two infinite series

$$a_0 + a_1 z + a_2 z^2 + \cdots \tag{6.48}$$

and

$$b_0 + b_1 z + b_2 z^2 + \cdots \tag{6.49}$$

which converge in an arbitrarily small finite interval toward the same function $f(z)$ then all the corresponding coefficients $a_k$ and $b_k$ of the two series must be the same:

$$a_k = b_k \tag{6.50}$$

The contradiction is only apparent if we realize that the quoted theorem assumes a limit-procedure which differs essentially from the way in which the interpolating series approximates. The two power series (6.48) and (6.49) have both *rigid* coefficients which do not change as the number of approximating terms increases. We merely add new and new terms to the sum without altering the partial sum we have obtained before. This is the way in which the Taylor-series approximates. But the series obtained by successive interpolations behaves

[18] See e.g. *Osgood*, Ref. 7, p. 134.

quite differently. Here the coefficients of the approximation are *permanently changing*, i.e. the coefficients $b_k$ of the approximation

$$\overline{f(x)} = b_0 + b_1 x + b_2 x^2 + \cdots b_n x^n \tag{6.51}$$

*are functions of* $n$:

$$b_k = b_k(n) \tag{6.52}$$

It is exactly the *flexible nature of the approximating coefficients*, adapted to the given range and to the number of terms by which we approximate, which is responsible for the possibility of extending the realm and efficiency of the Taylor-series in 3 directions:

1) to include functions which are *non-analytical*;

2) to *extend* the realm of convergence of the Taylor-series *outside* of the natural realm;

3) to *increase* the convergence of the Taylor-series *within* the natural realm.

We might expect some relation between the coefficients of the interpolating series and of the Taylor-series. The relation actually exists in form of an *asymptotic approach*: with increasing number of approximating terms *the coefficients* $b_k$ *converge monotonously toward the coefficients of the Taylor-series*:

$$\lim_{n \to \infty} b_k(n) = a_k \tag{6.53}$$

This relation holds, no matter whether the chosen range of the interpolating series is smaller, equal or larger than the realm of convergence of the Taylor series, as long as the point of expansion of the Taylor series is within or at the limit of the chosen range. The smaller the range the quicker the convergence. But the convergence is by no means "uniform." That means: if $\delta$ is the permitted difference between $b_k(n)$ and $a_k$, the $n$ necessary to reach this difference *depends on* $k$ and increases unlimitedly with increasing $k$. For a given definite value of $n$, however large it may be, there are always coefficients $b_k$ with sufficiently high index $k$ which differ from $a_k$ by a finite amount.[19]

2. **Economization of an asymptotic series.** Quite similar to the results obtained for the Taylor-series are the results obtainable for the asymptotic expansions. Taylor-series and asymptotic expansions are

[19] The proof of these statements follows from the consideration of the general interpolation formula of *Newton* for unequal intervals (see *Whittaker-Robinson*, Ref. 13, p. 24).

very similar in their nature—as mentioned before—since they are both extrapolating series. The only difference is that the asymptotic series makes use of the successive derivatives of the function $f(x)$ at a point where those derivatives do not exist in the complex sense because we are at a *point of singularity*. But they do exist if we approach the point *from one direction only*. We want to normalize this direction as the direction of the positive real axis. For example considering the function

$$y = e^{-\frac{1}{x^2}} \tag{6.54}$$

the point $x = 0$ is a point of singularity but all derivatives of the function exist at that point if we approach it from the positive $x$-axis.

Now the formal series

$$\overline{f(a+x)} = f(a) + f'(a)x + \frac{f''(a)}{2!}x^2 + \cdots \tag{6.55}$$

can again be considered as the limit of the Gregory-Newton interpolation formula, the points of interpolation being infinitely near to the point $x = 0$ and all toward the right from that point. The only difference is that the radius of convergence of the series is now zero. The radius of convergence of the Taylor-series being determined by the "nearest point of singularity," this radius converges toward zero if the point of expansion moves nearer and nearer to the point of singularity, although the coefficients of the Taylor-series may remain finite and approach definite limits. These limits are the coefficients of the asymptotic series.

Transforming the Taylor-series into an interpolating series we saw two advantages connected with such procedure: a) the convergence was quickened inside of the realm of convergence; b) the realm of convergence was extended outside of the original realm. Both advantages remain valid for asymptotic expansions although the first point has to be modified since these series are divergent from the beginning.

We choose an arbitrary finite range of the $x$-axis, limited at the left by the point of singularity $x = a$. The function shall be everywhere regular inside and at the limit of the range, even at the point $x = a$ if we only approach it along the $x$-axis, coming from the right. We then construct an approximating series, obtained by trigonometric interpolation in $n$ points. The interpolating series has the following two advantages, compared with the asymptotic series:

a) Using $n$ terms of the interpolating series the maximal error of the

approximation at any point of the range is smaller than the corresponding error by using $n$ terms of the asymptotic series.

b) This error converges monotonically toward zero if we go with $n$ higher and higher, while the error of the asymptotic series decreases at first and then increases again with increasing $n$.

We want to add that the same asymptotic relations we have observed between the coefficients of the Taylor-series and of the interpolating series, hold again between the coefficients of the asymptotic expansion and of the interpolating series:

$$\lim_{n\to\infty} b_k(n) = a_k \tag{6.56}$$

*The coefficients of the interpolating series converge monotonously toward the coefficients of the asymptotic expansion.* Again the speed of convergence depends on $k$ and decreases with increasing $k$.

*Example*: *Economization of the Stirling series.* One of the most important transcendental functions: the Gamma-function is defined by the following functional equation:

$$\Gamma(x+1) = x\Gamma(x) \tag{6.57}$$

together with the boundary condition:

$$\Gamma(1) = 1 \tag{6.58}$$

For large values of $x$ the function $\Gamma(x)$ can be asymptotically represented as follows:

$$\Gamma(x) = \sqrt{2\pi}\, e^{-x} x^{x-\frac{1}{2}} \tag{6.59}$$

Let us define the following analytical function:

$$\sigma(x) = \frac{\Gamma(x)}{\sqrt{2\pi}\, e^{-x} x^{x-\frac{1}{2}}} \tag{6.60}$$

This function has a singularity at the point $x = \infty$ of the complex plane, but no singularity occurs if we approach the point of infinity along the positive real axis.

We change the point of infinity $x = \infty$ into the point $x' = 0$ by the reciprocal transformation

$$x' = \frac{1}{x} \tag{6.61}$$

Then, interpolating $\rho(x')$ in the infinitesimal surroundings of $x' = 0$ we can expand $\rho(x')$ in a divergent power series of $x'$. The coefficients

of this series can be obtained with the help of the functional equation (6.57) which defines the Gamma-function. The expansion of $\sigma(x)$ has the form:

$$\sigma(x) = 1 + \frac{1}{12x} + \frac{1}{288x^2} - \frac{139}{51840}\frac{1}{x^3} + \cdots \tag{6.62}$$

This infinite series was the first historical example of an "asymptotic expansion" which does not satisfy the ordinary requirements of convergence for any value of $x$. The series was discovered by *Moivre* and *Stirling* and is usually called the "Stirling series."[20] Frequently the *logarithm* of the function $\sigma(x)$ is preferable to the function itself. We denote this new function by $B(x)$ and thus put:

$$B(x) = \log \Gamma(x) - (x - \tfrac{1}{2})\log x + x - \tfrac{1}{2}\log 2\pi \tag{6.63}$$

The functional equation which defines $B(x)$ is now:

$$B(x+1) - B(x) = 1 - (x + \tfrac{1}{2}) \log\left(1 + \frac{1}{x}\right) \tag{6.64}$$

together with the boundary condition

$$B(\infty) = 0 \tag{6.65}$$

The equation (6.64) is able to furnish the successive derivatives of $B(x')$ at the point $x' = 0$ which determine, according to the formula (1.2) of Gregory-Newton with infinitesimal $\epsilon$, the coefficients of the series-expansion of $B(x')$. We obtain

$$B(x) = \frac{1}{12}\frac{1}{x} - \frac{1}{360}\frac{1}{x^3} + \frac{1}{1260}\frac{1}{x^5} - \cdots \tag{6.66}$$

Generally

$$B(x) = \sum_{\alpha=1}^{\infty} (-1)^{\alpha-1} \frac{B_\alpha}{2\alpha(2\alpha-1)} \frac{1}{x^{2\alpha-1}} \tag{6.67}$$

where $B_\alpha$ are the so-called "Bernoulli-numbers."

Our program now is to transform the divergent asymptotic series (6.66) into a strictly convergent series by changing from extrapolation to interpolation. The range of the variable $x'$ shall coincide with the range (4.15); that means for $x$ the range

$$1 \leqq x \leqq \infty \tag{6.68}$$

[20] Concerning the history and theory of the Stirling series cf. *Davis*, Ref. 3, p. 179; *Whittaker-Watson*, Ref. 14, p. 251.

Although in chapter VIII a method will be shown by which the coefficients of the interpolating series could have been obtained directly on the basis of the functional equations (6.64), without knowing the actual course of the function $B(x)$, we prefer here the simple method of *numerical interpolation* in view of the fact that the numerical values of the Gamma-function are available with great accuracy in tabulated form. The table of *Davis*, Ref. 3, furnishes directly the Briggian logarithm of the Gamma-function to 10 decimal places at any point $x > 1$ provided that the necessary interpolations are applied. Hence the values of $B(x_k)$ can be calculated according to the formula (6.63) if we possess also a table of the ordinary logarithms to 10 decimal places.

The details of the numerical work are here omitted, only the general procedure shall be outlined. At first the points of interpolation $x_k$ have been determined with an accuracy of 10 decimal places, according to the formula

$$x_k = \frac{1}{1 + \cos \frac{\pi}{n} k} \tag{6.69}$$

For $n$ the value 12 has been adopted. The ordinates $y_k = B(x_k)$ (adding the factor $\frac{1}{2}$ for $y_0$ and $y_{12}$,) were obtained with the help of the 10-place tables of *Davis* and the 10-place logarithms of *Vega*. The last ordinate $y_{12}$ is zero. The two ordinates before: $y_{11}$ and $y_{10}$, were computed by using the asymptotic series (6.66).

The following table gives the points of interpolation and the corresponding ordinates:

| $k$ | $x_k$ | $y_k$ | |
|---|---|---|---|
| 0 | 1 | 0.0176022738 | |
| 1 | 1.0173323802 | 346326224 | |
| 2 | 1.0717967696 | 329489700 | |
| 3 | 1.1715728756 | 302499071 | |
| 4 | 4:3 | 266952556 | |
| 5 | 1.5887907066 | 225054534 | |
| 6 | 2 | 179540359 | (6.70) |
| 7 | 2.6983963724 | 133529154 | |
| 8 | 4 | 90292739 | |
| 9 | 6.8284271251 | 52963125 | |
| 10 | 1:0.0669872981 | 24239890 | |
| 11 | 1:0.0170370868 | 6165868 | |
| 12 | $\infty$ | 0 | |

These values of $y_k$ were entered in the formula (3.26) of the trigonometric interpolation and the computed coefficients follow:

$$\begin{aligned}
\tfrac{1}{2}C_0 &= 177756330.10^{-10} \\
C_1 &= 176235580 \\
C_2 &= -1758491 \\
C_3 &= -212360 \\
C_4 &= 25225 \\
C_5 &= -549 \\
C_6 &= -319 \\
C_7 &= 68 \\
C_8 &= -6 \\
C_9 &= 0 \\
C_{10} &= 0 \\
C_{11} &= 0 \\
C_{12} &= 0
\end{aligned}$$

The last digit of these figures ($10^{\text{th}}$ decimal place) may be unreliable to $\pm 1$ units.

For an exactness of $\pm 10^{-9}$ the last valuable coefficient $C_8$ may be omitted and the polynomial of $7^{\text{th}}$ order:

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1\left(\frac{1}{x}\right) + \cdots c_7 T_7\left(\frac{1}{x}\right) \tag{6.72}$$

formed. If we substitute for $T_k\left(\frac{1}{x}\right)$ their expressions according to the table (4.16), we obtain the following expansion:

$$\begin{aligned}
\overline{f(x)} = 10^{-10}\Big(6 &+ 361911590\,\frac{1}{x} + 6368\,\frac{1}{x^2} - 12084960\,\frac{1}{x^3} \\
&- 101248\,\frac{1}{x^4} + 4359680\,\frac{1}{x^5} - 2603008\,\frac{1}{x^6} + 557056\,\frac{1}{x^7}\Big)
\end{aligned} \tag{6.73}$$

This series represents the function $B(x)$ if logarithms of the base 10 are used, instead of natural logarithms of the base $e$. Multiplying our series by the factor

$$\frac{1}{M} = 2.3025850930 \tag{6.74}$$

we convert the Briggian logarithms to natural logarithms and obtain finally the following representation of the natural logarithm of the Gamma-function, valid between $x = 1$ and $x = \infty$:

$$\begin{aligned}\log \Gamma(x) = (x - \tfrac{1}{2}) \log x - x + \tfrac{1}{2} \log 2\pi + 0.0000000014 \\ + 0.833332232 \frac{1}{x} + 0.0000014663 \frac{1}{x^2} - 0.0027826649 \frac{1}{x^3} \\ - 0.0000233132 \frac{1}{x^4} + 0.0010038534 \frac{1}{x^5} - 0.0005993647 \frac{1}{x^6} \\ + 0.0001282669 \frac{1}{x^7}\end{aligned} \tag{6.75}$$

The following table shows the comparison of the coefficients of the economized series with the original asymptotic coefficients

| $k$ | A. S. | E. S. |
|---|---|---|
| 0 | 0 | 0.000000001 |
| 1 | 0.083333333 | 0.083333223 |
| 2 | 0 | 0.000001466 |
| 3 | −0.002777778 | −0.002782665 |
| 4 | 0 | −0.000023313 |
| 5 | 0.000793651 | 0.001003853 |
| 6 | 0 | −0.000599365 |
| 7 | −0.000595238 | 0.000128267 |

(6.76)

The optimum of the asymptotic series (6.66) for the range (6.68) is obtained by keeping the first 3 terms and dropping the rest:

$$\overline{B(x)} = \frac{1}{12} \frac{1}{x} - \frac{1}{360} \frac{1}{x^3} + \frac{1}{1260} \frac{1}{x^5} \tag{6.77}$$

with a maximum error of $-6.10^{-4}$. This limit cannot be improved. The economized series brings the error down to $\pm 2.10^{-9}$ by going until

the power $x^{-7}$ and going to higher and higher powers the error diminishes to arbitrarily small amounts.

A very close approximation is obtainable already by the abbreviated series of 4th order:

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1\left(\frac{1}{x}\right) + \cdots c_4 T_4\left(\frac{1}{x}\right) \tag{6.78}$$

If we rearrange this series again to an ordinary power series and change to natural logarithms we obtain the following approximation:

$$\overline{B(x)} = 0.08333272\,\frac{1}{x} + 0.00003715\,\frac{1}{x^2} - 0.00305164\,\frac{1}{x^3} + 0.00074346\,\frac{1}{x^4} \quad (\pm 2.10^{-7}) \tag{6.79}$$

The corresponding asymptotic development[21] with the coefficients

$$\frac{1}{12}, \qquad 0, \qquad -\frac{1}{360}, \qquad 0 \tag{6.80}$$

approximates with a maximal error of $8.10^{-4}$.

The present example illustrates all the characteristic properties of an interpolating approximation: 1) the error decreases indefinitely with increasing order $n$ of the approximating polynomial; 2) the error is always smaller than that of the corresponding portion of the asymptotic series. 3) The coefficients of the power expansion have no rigid values but are all the time changing with the order $n$; 4) they converge more and more to the coefficients of the asymptotic expansion, but for a given definite $n$ the coefficients of high order always differ by a finite amount from the asymptotic coefficients.

One more feature of the series (6.71) is of noticeable interest. We have used 13 points of interpolation in obtaining the 13 coefficients $c_k$ of the table (6.71). The last 4 coefficients came out as zero, within an accuracy of $10^{-10}$. Thus only the first 9 powers of $1/x$ (including the power zero) are employed in the representation of $f(x)$ with an accuracy of $10^{-10}$. Nine points of interpolation would have given already the coefficients $C_k$ of the table and any further increase of the number of interpolating points does not change the final values of $c_k$ within the given accuracy.

[21] "Corresponding" means: "the polynomial of the same order." If some of the coefficients of the asymptotic series vanish, *they have to be counted nevertheless*.

Now let us consider the infinite Fourier-series:

$$f(x) = \tfrac{1}{2}C_0 + C_1 T_1\left(\frac{1}{x}\right) + C_2 T_2\left(\frac{1}{x}\right) + \cdots \tag{6.81}$$

where the coefficients $C_k$ are determined by the definite integrals:

$$\begin{aligned} C_k &= \frac{2}{\pi}\int_0^{\pi} F(t)\cos kt\,dt \\ &= \frac{2}{\pi}\int_1^{\infty} f(x)T_k\left(\frac{1}{x}\right)\frac{dx}{x\sqrt{x-1}} \end{aligned} \tag{6.82}$$

Since the coefficients $c_k$, obtained by interpolation, do not change with increasing number of interpolating points, *they are already identical with the final Fourier-coefficients* $C_k$, toward which the $c_k$ always converge. This holds, of course, only within the accuracy of $10^{-10}$. In other words: *the difference between the interpolating coefficients* $c_k$ *and the Fourier coefficients* $C_k$ *is smaller than* $10^{-10}$.

Thus we have obtained, by interpolating in a small number of points, a series which is very accurate not only as far as its *sum* is concerned, but also as far as its *coefficients*, in comparison with the coefficients of the infinite series, are concerned. Similar conditions will always hold whenever it happens that the last coefficients of an interpolating series are practically neglectable.

## VII. Extension of the interpolating series to a complex domain

If we interpolate an empirical function we are only interested in the representation of the function within the given limits. But if we interpolate an *analytical* function, the question arises: can we extend the series, obtained by interpolating the function in a finite range of the real axis, to the surrounding *complex* domain?

We are not concerned here with the convergence properties of the *infinite* series to which the interpolating series converges if the number of interpolating points increases toward infinity. Practically more important is the problem to estimate the error of the interpolating series with a *finite* number of terms. We assume a given finite $n$ and consider the series:

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots c_{n-1}T_{n-1}(x) \tag{7.1}$$

obtained by the condition that function and approximation shall coincide at the $n$ zeros of the first neglected polynomial, i.e. $T_n(x)$. The error of the approximation shall be called

$$\eta(x) = f(x) - \overline{f(x)} \tag{7.2}$$

Our problem is to estimate $\eta(x)$ not only within the limits of interpolation but also for limited portions of the complex plane.

We restrict our considerations to the inside of a *circle* of the complex plane. The radius of this circle shall be normalized to 1 and its center shall coincide with the origin $z = 0$. This circle is shortly called the "unit-circle." We assume that the range of the given interpolation of $f(x)$ extends from $-1$ to $+1$, i.e. Tshebysheff's polynomials shall be normalized to the classical range

$$a = -1, \qquad b = 1. \tag{7.3}$$

Thus the problem may be formulated as follows. We have interpolated the function $f(x)$ along the diameter of a circle. How much is the error of the series if we extrapolate it within the area of that circle?

We want to deal only with a restricted class of analytical functions $f(z)$ which are analytical everywhere inside of the unit-circle. Singularities *on* the unit-circle may occur, but even here we do not permit any infinities of $f(z)$ if we approach those singularities from the *inside* of the circle. These conditions are frequently satisfied by those transcendental functions for which powerful approximations by polynomials are possible not only in the real but even in the complex range.

We start with Cauchy's fundamental integral formula

$$f(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{\zeta - z} \tag{7.4}$$

The region of integration shall be completely inside of the unit-circle and shall include the point $z$ and also all the zeros of $T_n(z)$.

We expand the function

$$\frac{1}{\zeta - z} \tag{7.5}$$

in Tshebysheff's polynomials, according to (6.30) and (6.31):

$$\frac{1}{\zeta - z} = \frac{T'_n(\zeta)}{nT_n(\zeta)} + \frac{2}{T_n(\zeta)} \sum_{\alpha=1}^{n-1} \frac{T'_{n-\alpha}(\zeta)}{n - \alpha} T_\alpha(z) + \frac{1}{T_n(\zeta)(\zeta - z)} T_n(z) \tag{7.6}$$

Substituting this expansion in (7.4) we obtain an approximating series of the form:

$$\overline{f(x)} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots c_{n-1} T_{n-1}(x) \tag{7.7}$$

where the $c_k$ are defined as follows:

$$c_k = \frac{2}{2\pi i} \int \frac{f(\zeta) T'_{n-k}(\zeta)}{(n-k) T_n(\zeta)} d\zeta \tag{7.8}$$

In addition we have the remainder

$$\eta(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{T_n(\zeta)(\zeta - z)} \cdot T_n(z) \tag{7.9}$$

We see that $f(z)$ and $\overline{f(z)}$ coincide at the $n$ zeros of $T_n(z)$ and thus the series (7.8) is identical with the series obtained by trigonometric interpolation.

In the integrals (7.8) we may integrate separately around each one of the $n$ poles:

$$T_n(x_\alpha) = 0 \qquad (\alpha = 1, 2, \cdots n) \tag{7.10}$$

Integrating around $x_\alpha$ we can put:

$$\int f(\zeta) \frac{T'_{n-k}(\zeta)}{T_n(\zeta)} d\zeta = \int f(\zeta) T'_{n-k}(\zeta) \frac{\zeta - x_\alpha}{T_n(\zeta)} \frac{d\zeta}{\zeta - x_\alpha} \tag{7.11}$$

which yields, remembering Cauchy's integral formula (7.4):

$$c_k = 2 \sum_{\alpha=1}^{n} \frac{T'_{n-k}(x_\alpha)}{(n-k) T'_n(x_\alpha)} f(x_\alpha) \tag{7.12}$$

The definition of Tshebysheff's polynomials was:

$$T_k(x) = \cos kt \tag{7.13}$$

where

$$x = \cos t \tag{7.14}$$

Thus

$$\frac{T'_{n-k}(x)}{T'_n(x)} = \frac{n-k}{n} \frac{\sin (n-k)t}{\sin nt} \tag{7.15}$$

Due to this relation we get

$$\frac{1}{n-k} \frac{T'_{n-k}(x_\alpha)}{T'_n(x_\alpha)} = \frac{1}{n} \frac{\sin (n-k)t_\alpha}{\sin nt_\alpha} = \frac{1}{n} \cos kt_\alpha = \frac{1}{n} T_k(x_\alpha) \tag{7.16}$$

which yields

$$c_k = \frac{2}{n} \sum_{s=1}^{n} f(x_s) T_k(x_s) \tag{7.17}$$

Hence we have obtained once more the fundamental formula (3.20) of the trigonometric interpolation, at present deduced by integration in the complex, on the basis of Cauchy's integral theorem.

Our specific problem, however, is to investigate the error (7.9) of the approximation. For this purpose the integral

$$\rho(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)\, d\zeta}{T_n(\zeta)(\zeta - z)} \tag{7.18}$$

has to be studied. The contour-integral (7.18) defines an analytical function $\rho(z)$ of the variable $z$ which remains finite everywhere inside and even on the periphery of the unit-circle. The maximum of $\rho(z)$ along the periphery of the unit-circle shall be denoted by $K$. Then due to a well known estimation of Cauchy;[22]

$$| \rho(z) | \leqq K \tag{7.19}$$

Thus we obtain for the error $\eta(z)$ the following estimation which holds everywhere inside and on the boundary of the unit-circle:

$$| \eta(z) | \leqq K | T_n(z) | \tag{7.20}$$

The estimation (7.20) is frequently sufficiently close, viz. if the condition is fulfilled that $| \rho(z) |$ shows no large variations inside of the unit-circle. Then the absolute value of the error $\eta(z)$ is essentially determined by the absolute value of $T_n(z)$.

The extension of a series, obtained by interpolation along the real axis, into the complex domain is by its very nature an *extrapolation*. Hence we cannot expect that the uniform distribution of errors, that we have experienced as a characteristic feature of the interpolating series, shall still hold.' The error has to increase the further we move away from the line of interpolation. Indeed: while $T_n(z)$ merely oscillates between $-1$ and $+1$ along the real axis, this is no longer true in the complex. We have generally

$$T_n(z) = \tfrac{1}{2}(e^{int} + e^{-int}) \tag{7.21}$$

where

$$e^{it} = z + \sqrt{1 - z^2} \tag{7.22}$$

[22] See Osgood, Ref. 7, p. 119.

The largest increase of $T_n(z)$ occurs along the imaginary axis and the maximum of $|T_n(z)|$ is assumed at the two poles $z = \pm i$. Here we have

$$\varepsilon'' = \pm i(1 + \sqrt{2}) \tag{7.23}$$

and we can put with good approximation:

$$|T_n(\pm i)| = \tfrac{1}{2}(1 + \sqrt{2})^n \tag{7.24}$$

We see that whereas the estimated maximum error of the *real range* is $K$ the estimated maximum error of the *complete unit circle* is larger in the ratio

$$\tfrac{1}{2}(1 + \sqrt{2})^n \tag{7.25}$$

Also the comparison with the Taylor series shows a different picture if we extend our considerations to the complex range. In chapter 4 we have reduced a given power series

$$y = a_0 + a_1 z + a_2 z^2 + \cdots a_n z^n \tag{7.26}$$

to the order $n - 1$ by expressing $z^n$ in lower powers with the help of Tshebysheff's polynomials. The error of the reduction is[23]

$$\eta = \frac{a_n}{2^{n-1}} T_n(z) \tag{7.27}$$

while the mere neglect of the term $a_n z^n$ causes the larger error

$$\eta' = a_n z^n \tag{7.28}$$

If we only take into account the real axis, the economized series is in advance by the factor $2^{n-1}$ compared with the Taylor series of the same number of terms. The favorable balance ceases to be valid if we include in the realm of approximation the entire unit-circle. The absolute maximum error of the Taylor-series is then still

$$|\eta'|_m = |a_n| \tag{7.29}$$

while the interpolating series yields now for the same quantity the larger value:

$$\begin{aligned} |\eta|_m &= |a_n| \left(\frac{1 + \sqrt{2}}{2}\right)^n \\ &= |a_n| \cdot 1.207^n, \end{aligned} \tag{7.30}$$

assumed at the two points $z = \pm i$.

[23] The denominator $2^{n-1}$ in comparison with the larger denominator $2^{2n-1}$ of formula (5.3) is due to the fact that the range of interpolation extends at present from $-1$ to $+1$ instead of 0 to $+1$ as before.

Nevertheless, in large portions of the unit-circle the balance is still in favor of the interpolating series and only in a small section around the two poles $z = \pm i$ is the Taylor series in advance. But even more important is the circumstance that frequently the Taylor-development is practically not available about the center of the range considered, while we may be able to produce without great efforts the interpolating series. In such cases we may well be reconciled with the somewhat too large error of the interpolating series around the two poles of the complex range.

*Example*: For the natural logarithm of the Gamma-function, the power series (6.75) has been found which approximates in the range (6.68) with a maximal error not exceeding $\pm 2.10^{-9}$. The "unit-circle" has now the diameter $x' = 0$ to $x' = 1$ where $x'$ is the reciprocal of $x$ (see (6.61)). Hence this circle has the center at $x' = \frac{1}{2}$ and the radius 1/2. For the original variable $x$ itself the corresponding range is the *complete complex half plane to the right of the line* $x = 1$. Although the error increases at worst

$$\tfrac{1}{2}(1 + \sqrt{2})^8 = 290$$

times we will still have the great accuracy of

$$|\eta|_m < 0.6 \cdot 10^{-6} \tag{7.31}$$

everywhere inside of the large complex range $x \geqq 1$, $y$ arbitrary. Thus merely replacing $x$ by the complex variable

$$z = x + iy \tag{7.32}$$

we obtain an approximation of $\log \Gamma(z)$ which holds in a large complex domain with an accuracy which is still remarkably high, although essentially reduced compared with the accuracy of the interpolation. Also the distribution of errors is no longer uniform.

We see that it is sometimes quite advantageous to compute a function in the real range with excessive accuracy. Frequently the values of a function are much more accessible along the real axis than anywhere else. In such cases we may reproduce the function in a suitably chosen real range by trigonometric interpolation, employing an exaggerated accuracy. Then we extrapolate the series to the complex and the remaining accuracy will still be satisfactory.

It has to be pointed out that such favorable conditions for the extension of the series into the complex only hold *if the convergence of the given interpolating series is sufficiently strong*. If the convergence of the

series is unsatisfactory this might mar considerably the extrapolation to the complex. But such series are of no practical use even in the real range. An interesting example for a series of this type is the function

$$f(x) = \frac{1}{1 + \epsilon - x} \tag{7.33}$$

where $\epsilon$ is a small positive quantity and the range of interpolation is between 0 and 1. The expansion of this function in Tshebysheff's polynomials has a very slow convergence. The error of the interpolating series converges toward zero only if we stay in the immediate neighborhood of the real axis. The Taylor-series is here by far superior as far as representation inside of the unit-circle is concerned.

## VIII. Methods of determining the coefficients of the trigonometric interpolation

1. **The numerical method.** If the function $f(x)$ is tabulated, the coefficients of the interpolating power series can be obtained by direct numerical computation. We use the formulae (3.26) of the trigonometric interpolation, obtaining at first the ordinates $y_k = f(x_k)$ at the characteristic points $x_k$ directly from the table, eventually after proper interpolations. In this way we determine the coefficients $c_k$ of the expansion

$$f(x) = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots \tfrac{1}{2}c_n T_n(x) \tag{8.1}$$

and then rearrange the series to an ordinary power series of $n^{\text{th}}$ order. An example of this procedure is the expansion (6.75) by which the logarithm of the Gamma-function has been approximated.

2. **Integration of a linear differential equation of first order with rational coefficients.** If the function $y = f(x)$ satisfies a linear differential equation with rational coefficients, the coefficients of the interpolating series may be obtained by purely *analytical* tools, without knowing the explicit values of $y$ at some points of the range. We wish to develop a method for the approximate solution of such differential equations by which the coefficients of the interpolating power series can be determined solely with the help of the differential equation.

We restrict our considerations to a linear differential equation of *first* order although the method is equally applicable to linear differential equations of higher order as it will be pointed out briefly in a later part of the present chapter.

The general type of the differential-equation we want to consider at present may be written as follows:

$$A(x)y' + B(x)y = C(x) \tag{8.2}$$

where $A(x)$, $B(x)$, $C(x)$ are some given polynomials of $x$. Before formulating the general procedure we want to explain the method on the basis of three simple examples. These examples are characteristic of a large class of problems, although the routine-work of solving the recursion formulae is particularly simple in these examples, in order to put the emphasis on the essential mathematical details of the procedure.

*Problem* 1. (*Economization of the Taylor-series.*) *In the range* 0 *to* 1 *the function* $e^{-x}$ *shall be developed in an economical power series.*

The function

$$y = e^{-x} \tag{8.3}$$

satisfies the following differential equation

$$y' + y = 0 \tag{8.4}$$

with the boundary condition

$$y(0) = 1 \tag{8.5}$$

If we expand $y$ in the infinite series

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \tag{8.6}$$

and put this series in the differential equation (8.4), we can obtain the coefficients of the expansion at once by equating the resulting coefficient of $x^{k-1}$ to zero. This yields the recursion formula

$$ka_k + a_{k-1} = 0 \tag{8.7}$$

which is solved by

$$a_k = \frac{(-1)^k}{k!}\, a_0 \tag{8.8}$$

while the boundary condition (8.5) gives:

$$a_0 = 1 \tag{8.9}$$

Now let us approximate $y$ by the finite power series

$$\bar{y} = a_0 + a_1 x + \cdots a_n x^n \tag{8.10}$$

We again put this development in the differential equation (8.4) and equate the resulting coefficient of $x^{k-1}$ to zero. This can be done until

finally we arrive at the *last* power $x^n$. Here we have to stop as the next higher coefficient $a_{n+1}$ is not at our disposal. But we still can say that the finite series (8.10) is the exact solution of a differential equation if we modify the given differential equation as follows:

$$y' + y = \tau x^n \tag{8.11}$$

where $\tau$ is a constant that we may adjust in any way we want.

We may generally say that instead of solving the given differential equation

$$D(y) = 0 \tag{8.12}$$

by an infinite series, we *modify* the differential equation by putting a suitable term on the right side of the equation:

$$D(\bar{y}) = \tau x^n \tag{8.13}$$

due to which modification the differential equation becomes solvable by a finite power-series.

This artificial modification of the given differential equation can be considered as an "error" which becomes neglectable if $\tau$ is sufficiently small. For the range (4.15) the error increases steadily from $x = 0$ and assumes for $x = 1$ the maximum value $\tau$. Now the error can be greatly reduced by reducing $x^n$ to lower powers according to the method discussed in Chapter V. A considerable portion of $x^n$ can be absorbed by the lower powers if we replace $x^n$ by the expansion (5.4). The error which still remains has the form

$$\frac{\tau}{2^{2n-1}} T_n(x) \tag{8.14}$$

and is greatly reduced due to the large factor $2^{2n-1}$. Moreover, it has an *oscillatory* character instead of the monotonically increasing function $\tau x^n$. The transformation of the right side of the differential equation from (8.13) to

$$D(\bar{y}) = \tau T_n(x) \tag{8.15}$$

has actually the effect that *the extrapolating power series is replaced by an interpolating power series of the same order.*

We apply the method to the given differential equation (8.4), modifying it in the form

$$y' + y = \tau T_n(x). \tag{8.16}$$

Let us denote the coefficients of the polynomial $T_n(x)$ by $B_k^n$:

$$T_n(x) = B_0^n + B_1^n x + \cdots B_n^n x^n \tag{8.17}$$

Then the recursion formulae which solve the differential equation (8.16) become

$$\begin{aligned} kb_k + b_{k-1} &= \tau B_{k-1}^n \qquad (k = 1, 2, \cdots n) \\ b_n &= \tau B_n^n \end{aligned} \tag{8.18}$$

These $n + 1$ equations are able to determine all the coefficients $b_k$ in terms of $\tau$. Finally the initial condition (8.9) yields the value of $\tau$. For example, let us approximate $y$ by a polynomial of 5th order. We take the coefficients $B_k^5$ out of the table (4.16) and obtain the following system of successive recursions, starting with $k = 5$ and going backwards.

$$\begin{aligned} b_5 &= \quad 512\tau \\ b_4 &= -1280\tau - 5b_5 = \quad -3840\tau \\ b_3 &= \quad 1120\tau - 4b_4 = \quad 16480\tau \\ b_2 &= \quad -400\tau - 3b_3 = -49840\tau \\ b_1 &= \quad 50\tau - 2b_2 = \quad 99730\tau \\ b_0 &= \quad -\tau - b_1 = -99731\tau \end{aligned} \tag{8.19}$$

The condition (8.9) yields:

$$\tau = -\frac{1}{99731} \tag{8.20}$$

and our series becomes:

$$\begin{aligned} e^{-x} &= \tfrac{1}{99731}(99731 - 99730x + 49840x^2 - 16480x^3 + 3840x^4 - 512x^5) \\ &= 1 - 0.9999900x + 0.4997443x^2 - 0.1652445x^3 + 0.0385036x^4 \\ &\qquad - 0.0051338x^5 \ (\pm 10^{-4}) \end{aligned} \tag{8.21}$$

The estimation of the error (see later) shows that the maximal error at any point of the range is approximately $\pm\frac{\tau}{8}$ which is $\pm 10^{-4}$.

The corresponding Taylor-expansion is as follows:

$$\begin{aligned} e^{-x} &= \tfrac{1}{120}(120 - 120x + 60x^2 - 20x^3 + 5x^4 - x^5) \\ &= 1 - x + 0.5x^2 - 0.16667x^3 + 0.04167x^4 - 0.00833x^5 \\ &\qquad (+1.4 \cdot 10^{-4}) \end{aligned} \tag{8.22}$$

At the end of the range: $x = 1$ the economical series yields the following rational approximation[24] of the constant $\frac{1}{e} = 0.36787944 \cdots$ :

$$\frac{1}{e} = \frac{39689}{99731} = 0.36787959 \text{ (error: } -0.00000015) \tag{8.23}$$

while the approximation of the Taylor-series is:

$$\frac{1}{e} = \frac{44}{120} = \frac{11}{30} = 0.3667 \text{ (error: } +0.0012) \tag{8.24}$$

An alternative treatment of the same method is possible with identical results but without needing the numerical values of $B_k^*$. We arrange our expansion from the very beginning in Tshebysheff's polynomials:

$$\bar{y} = \tfrac{1}{2}c_0 + c_1 T_1(x) + \cdots + c_n T_n(x) \tag{8.25}$$

and put this expansion in the modified differential equation, without converting $T_k(x)$ into a power series. Again we get recursion formulae for the $c_k$ which can be solved successively. For our present example and with $n = 5$ the recursion formulae become:

$$\begin{aligned} c_5 &= \tau \\ c_4 + 20c_5 &= 0 \\ c_3 + 16c_4 &= 0 \\ c_2 + 12c_3 + 20c_5 &= 0 \\ c_1 + 8c_2 + 16c_4 &= 0 \\ c_0 + 4c_1 + 12c_2 + 20c_5 &= 0 \end{aligned} \tag{8.26}$$

The solution is

$$\begin{aligned} c_5 &= \tau \\ c_4 &= -20\tau \\ c_3 &= 320\tau \\ c_2 &= -3860\tau \\ c_1 &= 31200\tau \\ c_0 &= -128660\tau \end{aligned} \tag{8.27}$$

[24] Whenever the polynomials $A(x)$, $B(x)$, $C(x)$ of the determining differential equation (8.2) have rational coefficients, the coefficients of the Taylor series or of the asymptotic expansion will likewise be rational numbers. The same holds of the new coefficients of the economized series. The common denominator of these coefficients, however, will be *much larger* than before.

The initial condition requires:

$$\tfrac{1}{2}c_0 - c_1 + c_2 - c_3 + c_4 - c_5 = 1 \tag{8.28}$$

and we obtain again the value (8.20) for $\tau$. The resulting series follows:

$$e^{-x} = \frac{1}{99731}\,(64330 - 31200T_1(x) + 3860T_2(x) - 320T_3(x) + 20T_4(x) - T_5(x)) \tag{8.29}$$

The arrangement of the economized power series in Tshebysheff's polynomials is theoretically much more satisfactory than the arrangement in powers of $x$. The natural functions of a trigonometric interpolation are the Fourier functions $\cos kt$ and these are, expressed in terms of $x$, the $T_k(x)$. Moreover, the convergence of the series can only be judged properly if the series is arranged in $T_k(x)$ and not in $x^k$. The coefficients $b_k$ of the power series are not characteristic and are unfit for any theoretical conclusions, due to their flexible nature which changes all the time with $n$. The coefficients $c_k$ of the orthogonal series change only slightly with increasing $n$ if the series has a rapid convergence. They are almost from the very beginning nearly equal to the $C_k$ of the final Fourier series toward which they converge as $n$ increases toward infinity.

On the other hand, for *practical* applications of the approximating series the arrangement in powers of $x$ is by far more convenient. Moreover, the recursion formulae which determine the coefficients of the series are frequently of a much simpler structure, and thus easier to solve, if the series is arranged in powers of $x$ instead of in Tshebysheff's polynomials $T_k(x)$.

*Problem* 2. (*Economization of an asymptotic expansion.*) The transcendental integral

$$E(x) = \int_x^\infty \frac{e^{-x}}{x}\,dx \tag{8.30}$$

is called the "logarithmic integral" of $x$. The following asymptotic expansion holds:

$$E(x) = \frac{e^{-x}}{x}\left(1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots\right) \tag{8.31}$$

*The expansion in the bracket shall be replaced by a convergent interpolating series valid for the range*

$$2 \leq x \leq \infty \tag{8.32}$$

By definition

$$E'(x) = -\frac{e^{-x}}{x} \tag{8.33}$$

Thus the function

$$y = xe^{x}E(x) \tag{8.34}$$

satisfies the following differential equation:

$$y' - \left(1 + \frac{1}{x}\right)y = -1 \tag{8.35}$$

We apply the reciprocal transformation

$$x' = \frac{1}{x} \tag{8.36}$$

and get in $x'$ the following differential equation, calling the new variable again simply $x$:

$$x^2y' + (1 + x)y = 1 \tag{8.37}$$

The new $x$ ranges between 0 and $\frac{1}{2}$. In order to adjust the polynomials $T_k(x)$ to the new limits, we have to use them in the form $T_k(2x)$.

We now solve the differential equation (8.37) exactly along the lines before. The asymptotic series would be obtained if we would correct the right side by a term of the form $\tau x^n$. We change this term, however, to $\tau T_n(2x)$ and set up our differential equation in the following form:

$$x^2y' + (1 + x)y = 1 + \tau T_n(2x) \tag{8.38}$$

This simple modification transforms the extrapolating asymptotic series to a strictly convergent interpolating series. The recursion formulae for the $b_k$ are now

$$\begin{aligned} kb_{k-1} + b_k &= \tau B_k^n \qquad (k = 1, 2, \cdots n) \\ b_0 &= 1 + \tau B_0^n \end{aligned} \tag{8.39}$$

where $B_k^n$ are the coefficients of $T_k(2x)$. They can be obtained with the help of the table (4.16), replacing $x$ by $2x$.

Let us choose $n = 8$ and solve the recursion formulae successively, starting with $k = 0$:

$$\begin{aligned}
b_0 &= 1 + \tau \\
b_1 &= -256\tau - b_0 = -1 - 257\tau \\
b_2 &= 10752\tau - 2b_1 = 2 + 11266\tau \\
b_3 &= -172032\tau - 3b_2 = -6 - 205830\tau \\
b_4 &= 1351680\tau - 4b_3 = 24 + 2175000\tau \\
b_5 &= -5767168\tau - 5b_4 = -120 - 16642168\tau \\
b_6 &= 13631488\tau - 6b_5 = 720 + 113484496\tau \\
b_7 &= -16777216\tau - 7b_6 = -5040 - 811168688\tau \\
b_8 &= 8388608\tau - 8b_7 = 40320 + 6497738112\tau = 0
\end{aligned} \tag{8.40}$$

The last condition determines $\tau$ to

$$\tau = -\frac{40320}{6497738112} = -0.620523624.10^{-5} \tag{8.41}$$

Putting back this value of $\tau$ in the expressions (8.40) we obtain the following expansion for the logarithmic integral function, valid for the range (8.32):

$$\begin{aligned}
E(x) = \frac{e^{-x}}{x}\Big(&0.9999938 - 0.9984053\,\frac{1}{x} + 1.9300918\,\frac{1}{x^2} \\
&- 4.7227762\,\frac{1}{x^3} + 10.5036112\,\frac{1}{x^4} - 16.7314160\,\frac{1}{x^5} \\
&+ 15.8018929\,\frac{1}{x^6} - 6.5066618\,\frac{1}{x^7}\Big)
\end{aligned} \tag{8.42}$$

The estimation of the error shows that the error of the expansion in the bracket is everywhere smaller than $\pm\tau$, i.e. $\pm 0.6.10^{-5}$. The estimated error of the asymptotic expansion (8.31) at the end $x = 2$ of the range is $+0.5$.

The function $E(x)$ is an important transcendental function not only in the real range but everywhere in the complex domain. The series (8.42) can be extended to the complex range, although with reduced accuracy, according to the principles discussed in Chapter VI. In large sections of the complex plane, to the right of the line $x = 2$, we

still get a rather high accuracy, with the exception of a limited region around the points $z = 2 \pm 2i$ where the error increases considerably.

Instead of trying to overstress the range of quick convergence of the series (8.42) we better proceed in a somewhat different manner. We divide the complex half plane (or even the full plane if necessary) in *sections* and set up the interpolating series independently, along a number of half rays. Then we can stay within reasonable limits concerning the extrapolation of each series and still the entire complex plane can be covered. For example, in the present problem of the logarithmic integral function, we would set up the interpolating series not only for the range $2 \leqq x \leqq \infty$ in reciprocal powers of $z = x + iy$, as we have done before, but we would repeat the process for the range $x = 0$, $2 \leqq y \leqq \infty$. These two series, together with an *ascending* power series obtained for the range $|z| = 0$ to $|z| = 3$ along the line $y = x$ (after taking out the logarithmic singularity at the point $z = 0$) permit a mutually interlocking extrapolation which covers the complete right half of the complex plane, without overstressing the capacity of either one of the interpolating series beyond their natural limitations.

Considering the great importance of the function along the *imaginary* axis (the "integral-sine" and "integral-cosine") we give here the result of the calculation for the interpolation between $y = 2$ and $y = \infty$ of the imaginary axis. The series is obtained exactly by the method before, except that the polynomials $T_k(x)$ have to be adjusted to the new range. This is done by replacing $x$ by $2xi$ so that the differential-equation to be solved is now

$$x^2y' + (1 + x)y = 1 + \tfrac{1}{2}T_n(2ix) \tag{8.43}$$

The result is the following set of coefficients which has to be substituted in place of the coefficients of the expansion (8.42) (inside of the bracket):

$$\begin{aligned}
b_0 &= \phantom{-}0.9999753 + 0.0000125\,i \\
b_1 &= -0.9967732 + 0.0063114\,i \\
b_2 &= \phantom{-}2.2591489 - 0.1471114\,i \\
b_3 &= -8.9292661 - 3.8083061\,i \\
b_4 &= \phantom{-}2.3270328 + 32.1403774\,i \\
b_5 &= \phantom{-}60.5020224 - 18.2377517\,i \\
b_6 &= -26.2787232 - 61.0795674\,i \\
b_7 &= -25.90257024 + 13.1158522\,i
\end{aligned} \tag{8.44}$$

The value of $\tau$ is:

$$\tau = (-2.47026159 + 1.25082513\, i)10^{-5} \tag{8.45}$$

and the absolute value of the error is at any point of the range smaller than $2.8 \cdot 10^{-5}$.

*Problem* 3. The following elementary problem is interesting owing to the extreme values of the asymptotic coefficients.

*In the range $x = 0$ to 1 the function*

$$y = -x \log x \tag{8.46}$$

*shall be expanded into an interpolating power series.*

The given function satisfies the following differential equation

$$\left(\frac{y}{x}\right)' = -\frac{1}{x} \tag{8.47}$$

which yields

$$xy' - y = -x \tag{8.48}$$

and we add the boundary condition

$$y(1) = 0 \tag{8.49}$$

The coefficients of the asymptotic series are now:

$$0, \infty, -\infty, \infty, \cdots \tag{8.50}$$

In accordance with the method employed in the previous examples we put again

$$xy' - y = -x + \tau T_n(x) \tag{8.51}$$

and obtain the following recursion formulae

$$(k-1)b_k = \tau B_k^n \qquad (k = 0, 2, 3, \cdots n)$$

$$0 = -1 + \tau B_1^n \tag{8.52}$$

The second condition determines $\tau$:

$$\tau = \frac{1}{B_1^n} \tag{8.53}$$

and thus our series becomes:

$$y = \frac{1}{B_1^n}\left(-B_0^n + B_1^n x^2 + B_2^n \frac{x^3}{2} + \cdots B_n^n \frac{x^n}{n-1}\right) + b_1 x \tag{8.54}$$

where $b_1$ has to be determined by the boundary condition (8.49).

For example for the choice $n = 10$ we obtain

$$\begin{aligned} y = {} & 0.005 + 4.1442\,x - 33\,x^2 + 211.2\,x^3 - 915.2\,x^4 \\ & + 256.256\,x^5 - 4659.2\,x^6 + 5461.3333\,x^7 - 3978.9714\,x^8 \\ & + 1638.4\,x^9 - 291.2711\,x^{10} \end{aligned} \qquad (8.55)$$

$(\pm 0.005)$

Although the coefficients of the series (8.55) show large variations and these variations increase with increasing $n$, each coefficient converging toward infinity, (with the only exception of $a_0$ which converges toward zero): the series approximates nevertheless closer and closer the given function $-x \log x$ at any point of the given range.

One might conjecture that the extreme values (8.50) of the asymptotic coefficients are responsible for the very slow convergence of the series. This is theoretically correct, if $n$ is growing toward infinity. But considering a *small* number of approximating terms, the conjecture does not hold that merely by inspecting the coefficients of the asymptotic series one may be able to tell something about the possible value of the maximal error $\eta_m$ of the interpolating series. In the case of the function $y = e^{-\frac{1}{x}}$ for example, the coefficients of the asymptotic expansion about $x = 0$ are all zero. And still the convergence of the interpolating series, if we expand again from 0 to 1, is rather slow for small values of $n$.

*Formulation of the general procedure.* After the foregoing examples we can easily outline the general procedure. We consider the solution of the linear differential equation of first order

$$D(y) = 0 \qquad (8.57)$$

where

$$D(y) = A(x)y' + B(x)y - C(x) \qquad (8.58)$$

The coefficients $A(x)$, $B(x)$, $C(x)$ are given polynomials of $x$, one of which may eventually be zero. $A(x)$ and $B(x)$ shall not vanish simultaneously at a point which belongs to the given range.

At first we normalize the given range to our standard range (4.15). This can always be done by a suitable linear transformation of $x$.

If we substitute for $y$ an infinite power series of the form

$$y = a_0 + a_1 x + a_2 x^2 + \cdots \qquad (8.59)$$

the coefficients of this expansion can be determined formally by letting $D$ operate on the infinite series (8.59) from term to term, then collect the terms of the same order and set each coefficient of the resulting sum

equal zero. The coefficients thus determined are the coefficients of the Taylor series if $A(0)$ is different from zero, and the coefficients of the asymptotic expansion if $A(0) = 0$.

We now put the *finite* series

$$\bar{y} = b_0 + b_1 x + \cdots b_n x^n \tag{8.60}$$

in the differential equation. We are able to modify the given differential equation (8.57) by suitable terms of the form $\tau x^m$ on the right side with the effect that the modified differential equation

$$D(\bar{y}) = \tau_0 x^n + \tau_1 x^{n+1} + \cdots \tau_m x^{n+m} \tag{8.61}$$

shall be solvable by the finite series (8.60). So far we have obtained the coefficients of a series which extrapolates $y$ in the neighbourhood of $x = 0$. We now change the character of the approximation from extrapolation to interpolation by replacing in the terms of correction $x^k$ by $T_k(x)$ so that the approximating differential equation (8.61) shall be put down in the following modified form:

$$D(\bar{y}) = \tau_0 T_n(x) + \tau_1 T_{n+1}(x) + \cdots \tau_m T_{n+m}(x) \tag{8.62}$$

Letting $D$ operate on the finite series (8.60) and comparing coefficients of the same order on both sides of the equation we get recursion formulae by which both the coefficients $b_k$ of the approximation and also the undetermined factors $\tau_i$ may be determined. The $\tau$-factors are decisive for estimating the error of the approximation $\bar{y}$, as it will be shown in the next part of the present chapter.

In order to avoid cumbersome numerical work it is frequently preferable to apply the correction of $D(y)$ in the following form:

$$D(\bar{y}) = T_n(x)(\tau_0 + \tau_1 x + \cdots \tau_m x^m) \tag{8.63}$$

which involves but *one* set of numerical coefficients $B_k^n$ in evaluating the coefficients of the approximation. On the other hand, we may arrange the approximation $\bar{y}$ in the form

$$\bar{y} = \tfrac{1}{2} c_0 + c_1 T_1(x) + \cdots c_n T_n(x) \tag{8.64}$$

and work consistently with the polynomials $T_k(x)$ instead of the powers $x^k$. For this purpose we have to express the polynomials $A(x)$, $B(x)$, $C(x)$ of the given differential equation in terms of $T_k(x)$ which can be done with the help of the table (5.6). Then we have to make use of the formula for the differentiation of Tshebysheff's polynomials:

$$T_k'(x) = 4k[T_{k-1}(x) + T_{k-3}(x) + \cdots] \tag{8.65}$$

and of the algebraic multiplication theorem (4.17). In this procedure the numerical values of the coefficients of $T_k(x)$ do not enter at all. The form (8.62) of the modified differential equation is now preferable to the form (8.63).

*Estimation of the error.* The principle of the above discussed approximation method is to modify the given differential equation (8.57) in the following manner:

$$D(\bar{y}) = \tau T_m(x) \tag{8.66}$$

Generally more than one correction term will be present. But it suffices to treat the problem for one term alone because for a sum of terms we apply the same principles for each term separately and then obtain the estimation of the resulting error as the absolute sum of the partial errors due to each term.

The error of the approximation $\bar{y}$ shall be called $\eta(x)$:

$$\eta(x) = y(x) - \overline{y(x)} \tag{8.67}$$

Since $D$ is a linear differential operator, we have

$$D_0(\eta) = D(y) - D(\bar{y}) = -\tau T_m(x) \tag{8.68}$$

$D_0$ is the notation for the *homogeneous* part of the differential operator (8.58), i.e.

$$D_0(\eta) = A(x)\eta' + B(x)\eta \tag{8.69}$$

For the solution of the inhomogeneous differential equation

$$D_0(\eta) = -\tau T_m(x) \tag{8.70}$$

we use the method of "variation of the constants". Let us denote by $u(x)$ the solution of the homogeneous equation:

$$D_0(u) = 0 \tag{8.71}$$

We now put

$$\eta(x) = \lambda(x)u(x) \tag{8.72}$$

and obtain

$$A(x)\lambda'(x)u(x) = -\tau T_m(x) \tag{8.73}$$

which yields

$$\lambda(x) = -\tau \int T_m(x)\alpha(x)\,dx \tag{8.74}$$

where we have denoted

$$\alpha(x) = \frac{1}{A(x)u(x)} \tag{8.75}$$

Let us suppose that we satisfy a given initial condition at the point $x = 0$. Then the error there is zero and we obtain:

$$\eta(x) = -\tau u(x) \int_0^x T_m(x)\alpha(x)\,dx \tag{8.76}$$

This expression of the error holds not only inside but also outside of the range of interpolation and thus allows us to estimate the error equally well for the eventual extrapolation of the series into the complex domain.

An estimation of the integral (8.76) is possible on the basis of Cauchy's inequality

$$\left|\int f(x)g(x)\,dx\right| \leqq \sqrt{\int |f(x)|^2\,dx \int |g(x)|^2\,dx} \tag{8.77}$$

But frequently this estimation would overrate the error considerably. In order to get a closer estimation we integrate at first by parts and thus separate one part of the integral that usually dominates compared with the remaining part. The justification of this procedure lies in the fact that $T_m(x)$ has an *oscillatory* character while $A(x)$ and $u(x)$ are comparatively smooth, at least if we go with $m$ high enough, and if we assume that $A(x)$ does not come too near to zero in the given range.

Now the integration of Tshebysheff's polynomials yields the following expression, proved easily by the basic definition of $T_k(x)$:

$$S_k(x) = \frac{1}{4}\left(\frac{T_{k+1}(x)}{k+1} - \frac{T_{k-1}(x)}{k-1}\right) - \frac{1}{2(k^2-1)} \tag{8.78}$$

We thus obtain by integrating (8.76) by parts:

$$\int_0^x T_m(x)\alpha(x)\,dx = S_m(x)\alpha(x) - \int_0^x S_m(x)\alpha'(x)\,dx \tag{8.79}$$

If $\alpha(x)$ is sufficiently smooth and finite, the first terms at the right of (8.79) is dominating. We can see that by integrating again by parts, which again brings $m+1$ in the denominator. We therefore get practically the estimation:

$$\eta(x) = -\tau\frac{S_m(x)}{A(x)} \tag{8.80}$$

Furthermore, inside of the range of interpolation we can put, considering the maximum value 1 of $T_k(x)$:

$$|S_m(x)| \leqq \frac{1}{4}\left(\frac{1}{m+1} + \frac{1}{m-1} + \frac{2}{m^2-1}\right) = \frac{1}{2}\frac{1}{m-1} \tag{8.81}$$

This yields

$$|\eta(x)| \leqq \frac{|\tau|}{2(m-1)}\frac{1}{|A(x)|} \tag{8.82}$$

This estimation breaks down if $A(x)$ gets too small or even zero in the given interval.

In the foregoing "Problem 1" the $A(x) = 1$ and we obtain for the complete interval 0 to 1:

$$|\eta(x)| \leqq \frac{\tau}{8} = 1.3 \cdot 10^{-6} \tag{8.83}$$

The form (8.80) of the error is a very convenient estimation also for the extension of the series to the complex if only $A(x)$ does not become too small. Naturally, the estimation (8.81) does not hold any more in the complex domain. Here we have to make use of the expression (8.78) and the corresponding maximum of $T_{m+1}(x)$.

The "Problem 2" is an example of the case when the estimation (8.80) breaks down because $A(x)$ has a zero inside of the range. We have at present: $A(x) = x^2$. Around the end $x = 1$ of the range the previous method of integrating by parts still holds good. But around the other end of the range: $x = 0$ the integration by parts has to be employed in the *opposite* direction, integrating the *second* factor $\alpha(x)$. Hence we cannot give here one single expression for the error $\eta(x)$ which would hold all over the range and at the same time would be close enough. It is easy to estimate for the complete range

$$|\eta| \leqq |\tau| \tag{8.84}$$

but this estimation overrates the error approximately 7 times around the point $x = 1$. The distribution of errors is in this problem much less uniform than in the problem before. In the next paragraph a method will be discussed by which in such cases the distribution of errors can be smoothed and the maximal error reduced.

*The method of excessive accuracy.* The estimation of the remainder of the interpolating approximation has to be accomplished with sufficient accuracy in order to be of any practical value. In mere convergence proofs the error of the finite series may be frequently overrated without

essential damage, because we do not care how many terms of the series are employed for a definite accuracy. But at present, being interested in a finite power series which shall represent a given function within a given range in the most economical way, i.e. with the smallest number of terms for a given accuracy, we are bound to estimate the error rather closely; if the estimated maximum is essentially larger than the actual maximum of the error, we cannot take full advantage of the approximation obtained. The estimation (8.82) was based on the "smoothness" of the function (8.75). But this smoothness is occasionally not sufficiently fulfilled. In such cases we may find it difficult to obtain a satisfactory estimation of the error. And also, the distribution of errors over the given range tends to be less uniform than it could be if the points of interpolation were distributed in a more adequate manner. Both difficulties may be eliminated by the following "method of excessive accuracy".

The coefficients $b_k$ of the interpolating series depend on the order $n$ of the approximating polynomial. Even if we arrange the series in Tshebysheff's polynomials, we still have the dependency of the $c_k$ on $n$, although much less pronounced. Thus we cannot evaluate the coefficients of the approximation with increasing $n$ *successively*, in the same manner as we can do it with the coefficients of the Taylor series. Each $n$ has to be treated independently, with the *full* set of coefficients. However, whereas we are unable to *increase* the $n$ without starting the evaluation of all of the $b_k$ coefficients over again, we can proceed in the *opposite* direction. We may *decompose* successively a given approximation, starting with a large value of $n$ and proceeding to smaller and smaller values. If the series is arranged in the $T_k(x)$, we merely drop successively the highest terms and thus reduce the series to a polynomial of smaller order and diminished accuracy. If the series is arranged in the powers $x^k$, we make use of the reduction process discussed in Chapter V.

This method of excessive accuray has the following 3 advantages:

a) an extrapolation of the series to the complex domain diminishes the accuracy of the series. Excessive accuracy along the real axis can guarantee satisfactory accuracy in the complex range;

b) the error of the series with an excessive number of terms is small compared with the error permitted. Hence even an overrated estimation of the error is not damaging, in view of the fact that we increase the error anyway by reducing the series to a smaller number of terms;

c) the reduction process *smoothes* the distribution of errors if the original distribution was not even enough.

Each step of the reduction process involves, of course, a new error. But we have *complete control* over these errors by knowing exactly the amount by which the maximal possible error increases if we reduce the series from $m$ to $m-1$; omitting the term $c_m T_m(x)$ we merely add to the previous maximal error the amount $|c_m|$.

*Example* 1. The series (8.42) for the logarithmic integral in the interval $x = 2$ to $\infty$ was obtained by employing an interpolating power series of $\frac{1}{x}$ or 7th order. If we employ in exactly the same manner a power series of 9th order, and then reduce it by two powers, we obtain the following approximation:

$$E(x) = \frac{e^{-x}}{x}\left(0.9999965 - \frac{0.9989710}{x} + \frac{1.9487646}{x^2} - \frac{4.9482092}{x^3} + \frac{11.7850792}{x^4} - \frac{20.4523840}{x^5} + \frac{21.1491469}{x^6} - \frac{9.5240410}{x^7}\right) \tag{8.85}$$

The maximal error of the series in the bracket is

$$|\eta_m| = 0.35 \cdot 10^{-6} \tag{8.86}$$

which is slightly over one half of the previous maximum.

*Example* 2. For almost all the fundamental transcendentals of mathematical physics and engineering the present approximation method yields new power expansions which are more efficient than the series so far existing. The range of the variable $x$ can frequently be divided into two sections: from $x = 0$ to a certain $x = a$ we expand in *ascending* powers: from $x = a$ to $x = \infty$ we expand in *descending* powers. The limit $x = a$ is arbitrary but we may choose it by the consideration that we shall not need too many powers of either series. The accuracy obtained depends, of course, on the number of powers employed. By decomposing a series of high accuracy we get a whole *spectrum* of power series of decreasing order $n$, and for any given degree of accuracy one may pick out the power series of the right number of terms.

The following example shows how one may tabulate such a spectrum of series starting with the order 5 and successively go down to the order 0. We have chosen the important transcendental $J_0(x)$ i.e. Bessel's function of the order 0. $J_0(x)$ can be represented within the range

$$1 \leqq x \leqq \infty \tag{8.87}$$

as the real part of the following expression:

$$Z_0(x) = \frac{e^{ix}}{\sqrt{\pi x}}(1-i)\left(b_0 + \frac{b_1}{x} + \frac{b_2}{x^2} + \cdots\right) \tag{8.88}$$

The coefficients $b_k$ are complex numbers and are listed in the table on page 191. The last column $\eta_m$ gives the absolute maximum of the error of the expansion in the bracket, at any point of the range. The first row ($n = \infty$) contains the coefficients of the asymptotic expansion; (the estimation $\eta_m$ connected with this row assumes that we employ but *two* terms of the series since otherwise the error increases again). All these series are included in one single expansion which is arranged in Tshebysheff's polynomials and where the successive reductions are obtained by merely dropping successively the last term of the series. The coefficients of this series, with the choice $n = 8$, were obtained as follows:

$$\begin{aligned}
c_0 &= \phantom{-}0.98336963 - 0.05217279\,i \\
c_1 &= -0.02049421 - 0.04806489\,i \\
c_2 &= -0.00296541 + 0.00420792\,i \\
c_3 &= \phantom{-}0.00079028 - 0.00004889\,i \\
c_4 &= -0.00011483 - 0.00010662\,i \\
c_5 &= \phantom{-}0.00000304 + 0.00003515\,i \\
c_6 &= \phantom{-}0.00000575 - 0.00000665\,i \\
c_7 &= -0.00000285 + 0.00000053\,i \\
c_8 &= \phantom{-}0.00000062 + 0.00000004\,i
\end{aligned} \tag{8.90}$$

$$(\eta_m = 1.5 \cdot 10^{-7})$$

3. **Linear differential equations of second and higher order, with rational coefficients.** The method of solving a linear differential equation with rational coefficients, as discussed in the previous point, can obviously be generalized, without essential modifications, to differential equations of higher than first order. Again a range of $x$ can be chosen, the solution set up in form of a finite power series, the correction terms established which render it possible to solve the differential equation in form of a finite series, then comes the transformation of the terms $x^k$ to $T_k(x)$ and finally the deduction of the recursion formulae which deter-

(8.89)

| $n$ | $b_0$ | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $\eta_n$ |
|---|---|---|---|---|---|---|---|
| $\infty$ | $1$ | $-\frac{i}{8}$ | $-\frac{9}{128}$ | $\frac{75}{1024}i$ | $\frac{3675}{32768}$ | $-\frac{297675\,i}{1310720}$ | 0.07 |
| 5 | $0.999991 + 0.000007\,i$ | $0.000771 - 0.125504\,i$ | $-0.081167 + 0.004889\,i$ | $0.057967 + 0.065103\,i$ | $-0.018530 - 0.058644\,i$ | $0.001558 + 0.017998\,i$ | $1.2 \cdot 10^{-5}$ |
| 4 | $0.999994 + 0.000042\,i$ | $0.000618 - 0.127261\,i$ | $-0.079950 + 0.018950\,i$ | $0.054558 + 0.025731\,i$ | $-0.014634 - 0.013648\,i$ | | $4.4 \cdot 10^{-5}$ |
| 3 | $1.00011 + 0.00015\,i$ | $-0.00304 - 0.13067\,i$ | $-0.06166 + 0.03601\,i$ | $0.02529 - 0.00156\,i$ | | | $2.10^{-4}$ |
| 2 | $1.0009 - 0.0001\,i$ | $-0.0113 + 0.1298\,i$ | $-0.0237 - 0.0337\,i$ | | | | $0.9 \cdot 10^{-3}$ |
| 1 | $1.004 - 0.004\,i$ | $-0.041 - 0.096\,i$ | | | | | $6.10^{-3}$ |
| 0 | $0.98 - 0.05\,i$ | | | | | | $6.10^{-2}$ |

mine the coefficients of the approximation, together with the $\tau$-factors of the correcting terms.

The formal application of the method is quite elementary but considerable care is necessary to *prepare* the given differential equation for a power series expansion. We have to remember that not all types of functions are suitable for an approximation in the large, if some accuracy is required and a prohibitive number of terms shall be avoided. Functions of an exponential type, or functions with many maxima and minima cannot be expanded efficiently in a power series if the requested range is large. Now differential equations with *constant* coefficients yield exactly this type of functions. Differential equations with *variable* coefficients have still the same type of solutions except that the *amplitudes* of those exponential (or periodic) solutions are no longer constant but functions of $x$. The purpose of the approximation is to expand not the complete solution but the *amplitude alone* in a power series. For example, solving Bessel's differential equation of the order zero in the range between $x = 1$ to $x = \infty$ we have separated from very beginning the factor

$$\frac{1}{\sqrt{x}} e^{ix} \tag{8.91}$$

and set up the differential equation only for the amplitude factor, using $\frac{1}{x}$ as the independent variable. We then obtain a descending power series in $x$ and the complete solution of the differential equation gets the form (8.88).

This example shows already that the general problem of solving differential equations on the basis of the present method is an extended study in itself which cannot be included in the present investigation. The fact shall only be mentioned that the differential equations of mathematical physics and of wave-mechanics can be treated successfully by the above discussed approximation method which also helps efficiently in determining the *characteristic values* of a differential equation of Schroedinger's type. In many problems the points $x = 0$ and $x = \infty$ are the essential singularities of the differential equation and one can expand about the point $x = 0$ in ascending and about the point $x = \infty$ in descending powers of $x$. The characteristic value of the differential equation can be obtained by joining continuously the function and its derivatives at the common end-point $x = a$ of the two ranges. Ordinarily the joining of those two series offers considerable

difficulties since both Taylor-series and asymptotic expansion converge rather badly at the point where they shall join. This difficulty is here eliminated. Since our series are of an interpolating and not of an extrapolating nature, the error does not increase rapidly as we move away from the point of expansion but oscillates all over the range with practically uniform amplitude. Hence the ascending and the descending series join without difficulty, in fact any *overlapping* of the two regions can be obtained, if wanted. The accuracy of a certain number of terms surpasses considerably the accuracy of either the Taylor series or the asymptotic expansion of the same number of terms. Moreover, the error of the descending power series converges steadily toward zero, while the asymptotic series is bound to a definite minimum of the error —depending on the point $x = a$ where we make use of the series—which cannot be improved.

In the problem of preparing a given differential equation to the power series expansion the original form of the differential equation has to be abandoned. The one differential equation of $n^{\text{th}}$ order has to be replaced by a simultaneous system of $n$ differential equations of first order, where the function and its first $n - 1$ derivatives are treated as independent variables. We get by far more efficient approximations if we do not differentiate the approximation of the function, increasing thereby the error considerably, but approximate independently the function and its first $n - 1$ derivatives. Thus the approximation problem of a differential equation is closely linked with a special form into which we can transform that differential equation, viz. the "canonical form," corresponding to the "canonical equations" of Hamilton.[25] The preparation of the differential equation to the application of a power series expansion is equivalent to the performance of a suitable linear canonical transformation of these equations. Further details have to be omitted at present and postponed to an independent investigation.

**4. The method of selected points.** The above discussed method for the solution of linear differential equations can be approached from a somewhat different angle which permits us to generalize the method to differential equations the coefficients of which are not necessarily rational functions of $x$. Also linear functional equations may be included in the scope of the method.

[25] See Ref. 6.

If we interpolate a given function by a power series of $n^{\text{th}}$ order, we have only $n + 1$ coefficients at our disposal, while the function represents an infinity of independent values. We thus cannot expect to meet the values of the function in more than $n + 1$ points. These $n + 1$ points may be distributed over the given range in an arbitrary manner, but the distribution of these points will determine uniquely the approximation together with the manner in which the error varies along the given range. The method of trigonometric interpolation distributes these points in a definite unequidistant fashion, in order to get the optimal conditions for reducing the maximal error at any point of the range. On the other hand, Taylor series and asymptotic expansion may be characterized by the fact that the points of interpolation are all crowded around one particular point of the range: the point of expansion.

A quite similar discussion is possible if an a priori unknown function $y = f(x)$ is characterized by a differential equation or a functional equation. Suppose we are able to approximate the function with sufficient accuracy by the finite power series

$$\bar{y} = b_0 + b_1 x + \cdots b_n x^n \tag{8.92}$$

Then, if we substitute for $y$ the approximation $\bar{y}$ in the given determining equation, we will not be able to satisfy that equation in more than $n$ points, since we have only the $n + 1$ constants at our disposal and one boundary condition has to be satisfied, if the differential equation is of first order. Generally the number of points in which the given equation can be exactly satisfied, will equal the number $n + 1$ of available constants, minus the number of boundary conditions. Again the choice of the points in which the determining equation is exactly satisfied, will decide the character of the approximation obtained. For example, if we choose all those points in the infinitesimal surroundings of one point $x = a$ of the range, then again the coefficients $b_k$ of the expansion (8.92) will become identical with the first $n + 1$ coefficients of the Taylor series or the asymptotic expansion. But we will obtain a series of interpolating character if the points in which the differential equation is satisfied are spread more evenly over the given range.

We refer to the differential equation (8.63). We have corrected the given differential equation (8.57) by a "right side" in order to adjust it to an approximation by a finite power series. The right side vanishes at the $n$ zeros of $T_n(x)$. Hence we may characterize the approximation obtained by solving the modified differential equation (8.63) as follows. We replace the function $y$ by the approximation (8.92) and evaluate

the coefficients $b_k$ of the approximation by establishing the given differential equation (8.57) at $n$ selected points, viz. at the $n$ zeros of Tshebysheff's polynomial $T_n(x)$. We thus get $n$ linear equations which, in connection with the given boundary condition which can be satisfied exactly, determine the $n + 1$ coefficients of the approximation.

This "method of selected points" for the interpolating approximation of a given differential equation yields exactly the same coefficients that we have obtained before by solving a set of recursion formulae. Naturally, we will prefer the recursion formulae to the solution of $n + 1$ simultaneous equations, each containing eventually all the $n + 1$ variables. But the importance of the new formulation of the method lies in the fact that it is no longer bound to differential equations with rational coefficients. Also differential equations with non-rational coefficients, or other functional equations may be included.

Instead of choosing the selected points in which we set up the differential equation in coincidence with the zeros of a certain $T_k(x)$ we may pursue the following procedure which yields eventually a more uniform distribution of errors and thus a lowering of the maximal error. We assume that the function $y = f(x)$ is approximated in the given range by trigonometric interpolation. Thus we set up $y$ in the following form:

$$y = b_0 + b_1 x + \cdots b_n x^n + \beta(x) T_{n+1}(x) \tag{8.93}$$

Then we can *estimate* the position of those points where the given differential or functional equation will be exactly satisfied if we replace $y$ by the approximating series on the right side of (8.93), omitting the last term. The estimation cannot be exact, of course, in view of the fact that the function $\beta(x)$ is not known. But we get a satisfactory estimation if we handle $\beta(x)$ for the purposes of the estimation as a *constant*. This is particularly justified if the Fourier expansion of $f(x)$ in Tshebysheff's polynomials has a sufficiently rapid convergence.

*Example.* In the approximation (6.75) of the logarithm of the Gamma-function a function $B(x)$ was introduced, defined by (6.63), which satisfies the functional equation (6.64), together with the boundary condition (6.65). Let us expand $B(x)$ in the finite series

$$\overline{B(x)} = \tfrac{1}{2}c_0 + c_1 T_1\left(\frac{1}{x}\right) + \cdots c_n T_n\left(\frac{1}{x}\right) \tag{8.94}$$

and obtain the coefficients of the expansion by the method of selected points. The following table shows the results of the application of the method, selecting $n = 1, 2, 3$ and 4 suitable points of the range $x = 1$

to $x = \infty$. The estimation of the position of these points occurred in accordance with the method described at the end of the last paragraph. For the sake of comparison, the exact values $C_k$ of the Fourier coefficients are likewise included, to 8 decimal places. With increasing $n$ the coefficients $c_k$ converge more and more toward the coefficients $C_k$. The absolute term $\frac{1}{2}c_0$ was obtained by satisfying the boundary condition (6.65).

(8.95)

| $n$ | $1/2\,c_0$ | $c_1$ | $c_3$ | $c_5$ | $c_7$ |
|---|---|---|---|---|---|
| 1 | 0.04071493 | 0.04071493 | | | |
| 2 | 0.04099956 | 0.04058583 | −0.00041373 | | |
| 3 | 0.04093568 | 0.04057940 | −0.00040491 | −0.00004863 | |
| 4 | 0.04092984 | 0.04057977 | −0.00040491 | −0.00004895 | 0.00000589 |
| $\infty$ | 0.04092991 | 0.04057974 | −0.00040491 | −0.00004890 | 0.00000581 |

It is of noticeable interest how quick the convergence is and how well the obtained approximation compares with the approximation by the Fourier series of the same number of terms. A handicap of the method is the circumstance that it seems not very easy to obtain a reliable estimation of the error, except by increasing the number $n$ of selected points and employing the "method of excessive accuracy," which was discussed before.

## Conclusion

In the theory of analytical functions the Taylor series plays a rôle of fundamental importance. The simple convergence behaviour of the series combined with the method of analytical continuation renders this series to an invaluable tool in studying the mathematical properties of analytical functions. For the practical representation of the function in a given range the Taylor series has the disadvantage of very slow convergence. This is due to the fact that the expansion utilizes the infinitesimal surroundings of one point of the complex plane and *extrapolates* the function everywhere else. The present investigation makes use of power series expansions of an *interpolating* nature. Such series are ordinarily employed only in connection with functions of a real variable. The present study intends to show how beneficial such

series are in approximating also analytical functions. The advantage of the interpolating series is its improved efficiency, being able to yield with the same number of terms a much more accurate approximation than the Taylor series. Moreover, the customary asymptotic expansions can be replaced by expansions of similar form but modified coefficients, expansions which converge toward the given function at any point of the chosen range and which have an improved accuracy compared with the asymptotic expansion of the same number of terms.

The coefficients of these expansions can be obtained by recursion formulae, similarly to the recursions by which the coefficients of the Taylor series or the asymptotic series are obtained, provided that the function satisfies a linear differential equation with rational coefficients. But the disadvantage of these coefficients is that they depend on the order $n$ of the approximating polynomial. It is difficult to obtain explicit expressions for the coefficients in their dependence on $k$, the order of the coefficient in the series, and on $n$, the order of the approximating polynomial. Another disadvantage of these series is their complicated convergence behaviour in the complex, compared with the simple convergence behaviour of the Taylor series. In the range of interpolation the convergence of the series is guaranteed by the general theory of orthogonal function systems, particularly the general theory of the Fourier series. But to find the exact realm of convergence of the series in the complex domain is generally quite a difficult problem.[26]

On the other hand, the purpose of these expansions is not to replace the infinite Taylor series by another infinite expansion. The practical importance of these series is the circumstance that already a small number of terms is able to yield a powerful approximation of the given function. Hence it is sufficient if the series can be produced without difficulty *for a given* $n$, and that is always possible. In view of this fact, it is in the nature of the problem that, instead of the general convergence investigation, we shall be interested in the *remainder* of the series for a given $n$. Actually a close estimation of the remainder is possible without great efforts, owing to the oscillatory nature of the error. Such an estimation can be given not only inside of the realm of interpolation but also for the extension of the series into the complex domain. Here the error naturally increases but still in extended regions of the unit-circle (although not in the entire unit-circle) the series can successfully compete with the Taylor series of the same number of terms.

[26] Consult on this problem *Walsh*, Ref. 12.

Considering the importance of these series for powerful approximations in the large, it seemed to be adequate to include a number of characteristic examples, giving the interpolations of some important transcendentals and comparing them with the customary Taylor or asymptotic expansions.[27]

These investigations grew out of a lecture course on "Orthogonal Function Systems" and another course on "Approximation Methods of Analysis," given at Purdue University in two successive years. The basic ideas of the method were developed in remembrance of a conversation with Prof. O. Szasz—himself a pioneer in the field of power series approximations,[28]—from whom, many years ago, the author learned about the amazingly high degree of "non-orthogonality" of the powers of $x$ due to which a high power of $x$ in a given range can almost completely be replaced by a proper combination of the lower powers. Various attempts have been made in different directions until finally the view point of *interpolation* has been adopted as probably the most adequate platform of discussion.

**Table of the More Frequent Notations**

$\bar{y}$ = approximation of $y$.

$a_k$ = coefficients of the Taylor series or asymptotic series.

$b_k$ = coefficients of the interpolating power series of finite order.

$c_k$ = coefficients of the same series if arranged in Tshebysheff's polynomials.

$C_k$ = coefficients of the infinite Fourier expansion in $T_k(x)$.

$T_k(x)$ = Tshebysheff's polynomials but mostly adapted to the range between $x = 0$ and $x = 1$.

$B_k^n$ = coefficient of $x^k$ in the expansion of $T_n(x)$.

$n$ = order of the approximating polynomial; (occasionally $n - 1$ instead of $n$).

$a, b$ = limits of the range of approximation.

[27] Many more applications have been made which could not be included in the present publication. The method is very useful for the precision tabulation of the higher transcendentals. A suitable subdivision of a large range into smaller intervals with readjusted coefficients permits us to obtain high accuracy with a small number of approximating powers.

[28] See Ref. 11.

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# ABRIDGED FORMULAS FOR THE SOLUTION OF DIFFERENTIAL EQUATIONS OF THE THEORY OF ELASTICITY BY MEANS OF INTEGRAL POLYNOMIALS

By S. A. Savin

## 1. Abridged denotations of integral polynomials and their derivatives

In many problems of the theory of elasticity its differential equations are solved by means of integral polynomials. With a view to simplify these solutions we may use abridged denotations for the integral polynomials, which were suggested by the author in 1923.[1]

Let us briefly denote the integral algebraic function $u$ of degree $\omega$ with the three independent variables $x, y, z$

$$f(x, y, z) = \sum_{0}^{\omega} D_{\alpha\epsilon\nu} x^{\alpha} y^{\epsilon} z^{\nu}, \tag{1}$$

where the summation is extended over all-possible combinations of the integral positive indices $\alpha, \epsilon, \nu$ between the limits 0 and $\omega$. Here the indices $\alpha, \epsilon, \nu$ as well as their sum take all the integral positive values from 0 to $\omega$ which may be expressed symbolically by the condition:

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega \tag{2}$$

Let us call the sum of the indices $\alpha, \epsilon, \nu$ of the constant coefficient $D_{\alpha\epsilon\nu}$, viz.—the quantity $\alpha + \epsilon + \nu$, the order of this coefficient.

Taking the partial derivative of the function (1) with respect to $x$ we obtain the integral polynomial of degree $\omega - 1$:

$$\frac{\partial f}{\partial x} = \sum \alpha \cdot D_{\alpha\epsilon\nu} x^{\alpha-1} y^{\epsilon} z^{\nu}.$$

The exponents of the degree $\alpha - 1, \epsilon, \nu$ in this polynomial are connected by the condition:

$$0 \leq [\alpha - 1, \epsilon, \nu, (\alpha - 1) + \epsilon + \nu] \leq \omega - 1.$$

The introduction of new denotations instead of $\alpha, \epsilon, \nu$ shall not alter the final result of summing up the terms under the sign $\sum$, provided they satisfy the last condition. Therefore, replacing the index $\alpha$ under

[1] Savine, S. A., Ouvrages sur la théorie de l'élasticité, fasc. 1, article 1.



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the sign of the sum by the new denotation $\alpha_0 + 1$ the partial derivative of the function $f(x, y, z)$ may be presented as:

$$\frac{\partial f}{\partial x} = \sum_{0}^{\omega-1} (\alpha_0 + 1) \cdot D_{(\alpha_0+1),\epsilon,\nu} x^{\alpha_0} y^{\epsilon} z^{\nu}.$$

Returning from $\alpha_0$ to the former denotation of the index $\alpha$ and adjoining analogous expressions for $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ we obtain the formulas for partial derivatives of the first order from the function (1):

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \sum_{0}^{\omega-1} (\alpha + 1) D_{(\alpha+1),\epsilon,\nu} x^{\alpha} y^{\epsilon} z^{\nu} \\ \frac{\partial f}{\partial y} &= \sum_{0}^{\omega-1} (\epsilon + 1) D_{\alpha,(\epsilon+1),\nu} x^{\alpha} y^{\epsilon} z^{\nu} \\ \frac{\partial f}{\partial z} &= \sum_{0}^{\omega-1} (\nu + 1) D_{\alpha,\epsilon,(\nu+1)} x^{\alpha} y^{\epsilon} z^{\nu} \end{aligned}\right\} \tag{3}$$

where the integral positive indices $\alpha$, $\epsilon$, $\nu$ and their sum $\alpha + \epsilon + \nu$ are already connected by a new condition:

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 1. \tag{4}$$

In a similar way we obtain from the formula (3) the expressions of partial derivatives of the second order from the function (1):

$$\left.\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \sum_{0}^{\omega-2} (\alpha + 1)(\alpha + 2) D_{(\alpha+2),\epsilon,\nu} x^{\alpha} y^{\epsilon} z^{\nu} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial^2 f}{\partial x \partial y} &= \sum_{0}^{\omega-2} (\alpha + 1)(\epsilon + 1) D_{(\alpha+1),(\epsilon+1),\nu} x^{\alpha} y^{\epsilon} z^{\nu} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{5}$$

in which the integral positive indices $\alpha$, $\epsilon$, $\nu$ are connected by the condition:

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 2. \tag{6}$$

In a general case, as it is known, the number of terms of the integral polynomial (1) of degree $\omega$ with three variables is defined by the formula:

$$\frac{(\omega + 3)(\omega + 2)(\omega + 1)}{1 \cdot 2 \cdot 3}. \tag{7}$$

Consequently, in a general case each formula of the system (3) gives the number of terms:

$$\frac{(\omega+2)(\omega+1)\omega}{1\cdot 2\cdot 3}, \tag{8}$$

and each formula of the system (5)—the number of terms:

$$\frac{(\omega+1)\omega(\omega-1)}{1\cdot 2\cdot 3}. \tag{9}$$

Hence it is seen that the abridged denotations (1), (3), (5) considerably simplify the formulas.

### 2. Condition for the coefficients of a harmonic polynomial

Having the abridged formulas (5) it is easy to obtain in a general form the relation of the coefficients of the harmonic polynomial (1), i.e. when it satisfies Laplace's harmonic equation with three independent variables $x, y, z$:

$$\nabla^2 f(x,y,z) \equiv \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0. \tag{10}$$

By the formulas (5) we find the expression of the Laplacian of the function $f(x, y, z)$:

$$\begin{aligned}\nabla^2 f(x,y,z) = \sum_0^{\omega-2} x^\alpha y^\epsilon z^\nu \cdot [&(\alpha+1)(\alpha+2)D_{(\alpha+2)\epsilon\nu} \\ &+ (\epsilon+1)(\epsilon+2)D_{\alpha(\epsilon+2)\nu} + (\nu+1)(\nu+2)D_{\alpha\epsilon(\nu+2)}].\end{aligned} \tag{11}$$

In order that this expression with the variables $x, y, z$ be equal to zero the following condition is necessary:

$$\begin{aligned}(\alpha+1)(\alpha+2)D_{(\alpha+2)\epsilon\nu} + (\epsilon+1)(\epsilon+2)D_{\alpha(\epsilon+2)\nu} \\ + (\nu+1)(\nu+2)D_{\alpha\epsilon(\nu+2)} = 0.\end{aligned} \tag{12}$$

This equation must be applied for all the integral positive values of the indices $\alpha, \epsilon, \nu$ and their sum $\alpha + \epsilon + \nu$ according to the condition (6). The number of the equations of the type (12) for an integral polynomial of degree $\omega$ is defined by the formula (9).

### 3. Condition for the coefficients of a biharmonic polynomial

Another important equation in the theory of elasticity is the biharmonic equation:

$$\nabla^2\nabla^2 f(x,y,z) = 0, \tag{13}$$

where the symbol of the Laplacian $\nabla^2$ denotes as before the sum of the second derivatives of the function $\nabla^2 f(x, y, z)$, i.e.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{14}$$

Taking from the function (11) the partial derivative of the second order with respect to $x$ we obtain by the formulas (5) of the differentiation of the polynomials:

$$\frac{\partial^2}{\partial x^2}\nabla^2 f(x, y, z) = \sum_0^{\omega-4} (\alpha + 1)(\alpha + 2)[(\alpha + 3)(\alpha + 4)D_{(\alpha+4)\nu\rho}$$
$$+ (\epsilon + 1)(\epsilon + 2)D_{(\alpha+2)(\epsilon+2)\rho} + (\nu + 1)(\nu + 2)D_{(\alpha+2)\epsilon(\nu+2)}]x^\alpha y^\epsilon z^\nu.$$

Analogous formulas may be obtained from (11) also for the partial derivatives:

$$\frac{\partial^2}{\partial y^2}\nabla^2 f, \qquad \frac{\partial^2}{\partial z^2}\nabla^2 f.$$

Summing up these three equalities and knowing that the biharmonic equation (13) is to be satisfied for all-possible values of $x, y, z$ we obtain a dependence among the coefficients of the biharmonic polynomial (1):

$$\begin{aligned}
&(\alpha + 1)(\alpha + 2)(\alpha + 3)(\alpha + 4)D_{(\alpha+4)\nu\rho} \\
&\qquad + (\epsilon + 1)(\epsilon + 2)(\epsilon + 3)(\epsilon + 4)D_{\alpha(\epsilon+4)\rho} \\
&+ (\nu + 1)(\nu + 2)(\nu + 3)(\nu + 4)D_{\alpha\epsilon(\nu+4)} \\
&\qquad + 2(\alpha + 1)(\alpha + 2)(\epsilon + 1)(\epsilon + 2)D_{(\alpha+2)(\epsilon+2)\rho} \\
&\quad + 2(\epsilon + 1)(\epsilon + 2)(\nu + 1)(\nu + 2)D_{\alpha(\epsilon+2)(\nu+2)} \\
&\qquad + 2(\nu + 1)(\nu + 2)(\alpha + 1)(\alpha + 2)D_{(\alpha+2)\epsilon(\nu+2)} = 0,
\end{aligned} \tag{15}$$

i.e. of an integral polynomial satisfying the equation (13).

The equation (15) is to be applied for all-possible values of the integral positive numbers $\alpha, \epsilon, \nu$, subordinate to the condition:

$$0 \leq [\alpha, \epsilon, \nu, \alpha + \epsilon + \nu] \leq \omega - 4. \tag{16}$$

Therefore the number of the equations of the type (15) for the integral polynomial (1) of degree $\omega$, according to the formula (7), is equal to:

$$\frac{(\omega - 1)(\omega - 2)(\omega - 3)}{1 \cdot 2 \cdot 3}.$$

## 4. Application of integral polynomials for the solution of the equations of the internal equilibrium of an isotropic body

Let us suppose that the expressions of the displacements $u$, $v$, $w$ of an isotropic body are given under the form of integral polynomials of degree $\omega$:

$$u = \sum_{0}^{\omega} A_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma}, \qquad v = \sum_{0}^{\omega} B_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma}, \qquad w = \sum_{0}^{\omega} C_{\alpha\beta\gamma} x^{\alpha} y^{\beta} z^{\gamma}. \tag{17}$$

Denoting the cubic dilatation in a current point of a body by $\Delta$ we obtain from (17) according to the formulas of differentiation (3):

$$\Delta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \sum_{0}^{\omega-1} x^{\alpha} y^{\beta} z^{\gamma} [(\alpha + 1) A_{(\alpha+1)\beta\gamma} + (\beta + 1) B_{\alpha(\beta+1)\gamma} + (\gamma + 1) C_{\alpha\beta(\gamma+1)}]. \tag{18}$$

If the volume force is but the own weight of an isotropic body then its displacements must satisfy, as it is known, the equations of the internal equilibrium of Lamé:[2]

$$\left.\begin{aligned} (\lambda + \mu) \frac{\partial \Delta}{\partial x} + \mu \nabla^2 u &= -q \cos(q, x) \\ (\lambda + \mu) \frac{\partial \Delta}{\partial y} + \mu \nabla^2 v &= -q \cos(q, y) \\ (\lambda + \mu) \frac{\partial \Delta}{\partial z} + \mu \nabla^2 w &= -q \cos(q, z) \end{aligned}\right\} \tag{19}$$

where:

$\lambda$, $\mu$—are the coefficients of Lamé's elasticity,

$q$—is the weight of the body per unit volume.

The own weight of the body in the differential equations (19) is a constant. Therefore, solving these equations by means of integral polynomials, the influence of the own weight may be considered separately according to the principle of superposition.

Therefore let us consider firstly the case when the volume force is absent, viz. when

$$q = 0. \tag{20}$$

Forming a partial derivative with respect to $x$ from the function (18) and taking according to the formula (11) the Laplacian $\nabla^2 u$, we put

[2] Poincairé, H. Leçons sur la théorie de l'élasticité, Paris, 1892, p. 86.

them into the left hand member of the equation I (19) and then, after reduction, we obtain:

$$(\lambda+\mu)\frac{\partial\Delta}{\partial x}+\mu\nabla^2 u=\sum_{0}^{\omega-2}x^\alpha y^\epsilon z^\nu\{(\lambda+\mu)(\alpha+1)[(\nu+1)C_{(\alpha+1)\epsilon(\nu+1)}+(\epsilon+1)B_{(\alpha+1)(\epsilon+1)\nu}]+(\lambda+2\mu)(\alpha+1)(\alpha+2)A_{(\alpha+2)\epsilon\nu}+\mu[(\epsilon+1)(\epsilon+2)A_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)A_{\alpha\epsilon(\nu+2)}]\}.$$

Hence it is seen that in the above case (20) the equation I (19) shall be satisfied for the current point $(x, y, z)$ of the body only on the condition:

$$(\lambda+\mu)(\alpha+1)[(\epsilon+1)B_{(\alpha+1)(\epsilon+1)\nu}+(\nu+1)C_{(\alpha+1)\epsilon(\nu+1)}]+(\lambda+2\mu)(\alpha+1)(\alpha+2)A_{(\alpha+2)\epsilon\nu}+\mu[(\epsilon+1)(\epsilon+2)A_{\alpha(\epsilon+2)\nu}+(\nu+1)(\nu+2)A_{\alpha\epsilon(\nu+2)}]=0. \quad (21)$$

In the same way we obtain the second condition from the equation II (19):

$$(\lambda+\mu)(\epsilon+1)[(\nu+1)C_{\alpha(\epsilon+1)(\nu+1)}+(\alpha+1)A_{(\alpha+1)(\epsilon+1)\nu}]+(\lambda+2\mu)(\epsilon+1)(\epsilon+2)B_{\alpha(\epsilon+2)\nu}+\mu[(\nu+1)(\nu+2)B_{\alpha\epsilon(\nu+2)}+(\alpha+1)(\alpha+2)B_{(\alpha+2)\epsilon\nu}]=0, \quad (22)$$

and the third condition from III (19):

$$(\lambda+\mu)(\nu+1)[(\alpha+1)A_{(\alpha+1)\epsilon(\nu+1)}+(\epsilon+1)B_{\alpha(\epsilon+1)(\nu+1)}]+(\lambda+2\mu)(\nu+1)(\nu+2)C_{\alpha\epsilon(\nu+2)}+\mu[(\alpha+1)(\alpha+2)C_{(\alpha+2)\epsilon\nu}+(\epsilon+1)(\epsilon+2)C_{\alpha(\epsilon+2)\nu}]=0. \quad (23)$$

From comparing the formulas (21)–(23) we can see that each following one may be obtained from the preceding by means of the circular permutation of the coefficients $A$, $B$, $C$ and indices $\alpha$, $\epsilon$, $\nu$.

The equations (21)–(23) must be applied for all-possible combinations of the integral positive indices $\alpha$, $\epsilon$, $\nu$ connected by the condition (6). Thus, the total number of the equations for the displacements (17), according to the formula (9), is equal to:

$$\tfrac{1}{2}(\omega+1)\omega(\omega-1).$$

It is seen from the equations (19) that the own weight of the body has a direct influence on the coefficients of displacements only of the second order.

Applying integral polynomials to the solution of the equations (19), when $q \neq 0$, we obtain the three new relations instead of the equations (21)–(23) for the coefficients of displacements of the second order:

$$\left.\begin{aligned}
&(\lambda+\mu)(B_{110}+C_{101})+2(\lambda+2\mu)A_{200}+2\mu(A_{020}+A_{002})\\
&\qquad\qquad +q\cos(q,x)=0,\\
&(\lambda+\mu)(C_{011}+A_{110})+2(\lambda+2\mu)B_{020}+2\mu(B_{002}+B_{200})\\
&\qquad\qquad +q\cos(q,y)=0,\\
&(\lambda+\mu)(A_{101}+B_{011})+2(\lambda+2\mu)C_{002}+2\mu(C_{200}+C_{020})\\
&\qquad\qquad +q\cos(q,z)=0.
\end{aligned}\right\} \tag{24}$$

For the solution of the equations (19) with the right-hand members, i.e. when the influence of the own weight of body is not separated, it is necessary to form according to the conditions (21)–(23) a system of equations for all the coefficients $A$, $B$, $C$ of the displacements, except the coefficients of the second order, which are defined separately according to the system (24).

The system of linear equations thus formed evidently cannot determine all the coefficients of displacements. In order to define them completely it is necessary to add more equations formed according to the conditions for the external surfaces of the body and its points of support.

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# A PROOF OF A THEOREM OF CARLEMAN*

By Benjamin Epstein

In this paper a proof is given of a theorem of Carleman regarding the zeros of an analytic function in a half-plane. An extension is made to the case of meromorphic functions and the function $f(z)$ which is used, is assumed throughout to be meromorphic everywhere in the finite plane. Essentially the theorem is proved by obtaining the mapping-function of a large half-circle on the unit circle and using the logarithm of the absolute value of this function as the Green's function for the problem. As the basis of the proof we use a theorem from potential-theory which is here put in a form suitable for function-theoretic applications. If the contour $\Gamma$ is made up of one or several arcs of sufficient regularity and if $G(z, \zeta)$ is the Green's function for the region[1] and if $f(\zeta)$ is regular everywhere in a certain closed region $G$ of the complex plane except possibly at a finite number of points, then the formula:

$$(1) \qquad \log |f(\zeta)| = \frac{1}{2\pi} \int_\Gamma u(\zeta) \frac{dG(z, \zeta)}{dn} ds - \sum_i G(a_i, z) + \sum_j G(b_j, z)$$

is valid. Here the $a_i$ are the affixes of the zeros and the $b_j$ are the affixes of the poles.[2]

We take as our mapping function the function

$$(2) \qquad w(\zeta) = \frac{\zeta - z}{\rho^2 - \bar{z}\zeta} \cdot \frac{\rho^2 + \zeta z}{\zeta + \bar{z}}$$

which maps the half-circle

$$Re(\zeta) \geq 0, \qquad |\zeta| \leq \rho$$

* This is part of a thesis submitted for the degree of Master of Science in Mathematics at the Massachusetts Institute of Technology.

[1] $G(z, \zeta)$ is a function, with $z$ fixed, $\zeta$ varying, regular in the whole region except at the point $P$. In the neighborhood of $P$ let the function become infinite like $\log \frac{1}{r}$ where $r = |z - \zeta|$ that is to say $G(z, \zeta) - \log \frac{1}{r}$ is analytic at $P$. Finally, we want $G(z, \zeta)$ to be zero on the contour $\Gamma$.

[2] L. Bieberbach—Lehrbuch der Funktionentheorie II, 1931. Chapter I. Section 6. Pp. 34 and 35.

P. Montel—Les Fonctions Entières ou Méromorphes, 1932. Chapter I.

on the unit circle, and so the Green's function becomes

$$-\log\left|\frac{\zeta - z}{\rho^2 - \zeta\bar{z}}\cdot\frac{\rho^2 + \zeta z}{\zeta + \bar{z}}\right|. \tag{3}$$

Remembering that $\zeta = \rho e^{i\phi}$ and $z = re^{i\phi}$ ($\zeta$ is our variable) we find that

$$\frac{\partial g}{\partial n} = \frac{4r(\rho^2 - r^2)\cos\theta\cos\phi}{[\rho^2 + r^2 - 2r\rho\cos(\theta - \phi)][\rho^2 + r^2 + 2r\rho\cos(\theta + \phi)]}. \tag{4}$$

It should be remembered that $\dfrac{\partial g}{\partial n} = -\dfrac{\partial g}{\partial r}$ on the large half-circle, and on the boundary $x = 0$ we take our derivative *normal* to the $y$-axis coming in from the left. What we want, then, is:

$$\begin{aligned}&-\frac{\partial}{\partial\mu}[\log(\mu + i\eta - z) - \log(\rho^2 - \bar{z}(\mu + i\eta))\\ &\qquad + \log(\rho^2 + (\mu + i\eta)z) - \log(\mu + i\eta + \bar{z})]_{\mu=0}\\ &= -\left[\frac{1}{i\eta - z} + \frac{\bar{z}}{\rho^2 - i\bar{z}\eta} + \frac{z}{\rho^2 + i\eta z} - \frac{1}{i\eta + \bar{z}}\right] \qquad (5)\\ &= -\left[-\frac{2\operatorname{Re} z}{x^2 + (y - \eta)^2} + \frac{2\rho^2\operatorname{Re} z}{(\rho^2 - \eta y)^2 + \eta^2 x^2}\right].\end{aligned}$$

Also terms for poles and zeros (in our case only zeros) are represented as:

$$-\sum_\nu \log\left[\frac{z - r_\nu e^{i\theta_\nu}}{\rho^2 - r_\nu e^{-i\theta_\nu} z_\nu}\cdot\frac{\rho^2 + r_\nu e^{i\theta_\nu} z}{z + r_\nu e^{-i\theta_\nu}}\right]. \tag{6}$$

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}\frac{4r(\rho^2 - r^2)\cos\theta\cos\phi\log|f(\zeta)|\rho\,d\theta}{(\rho^2 + r^2 - 2r\rho\cos(\theta - \phi))(\rho^2 + r^2 + 2r\rho\cos(\theta + \phi)}\\ &+\frac{1}{2\pi}\left\{\int_\tau^\rho + \int_{-\tau}^{\tau} + \int_{-\rho}^{-\tau}\right\}\log|f(i\eta)|\left[\frac{2\rho^2 x}{(\rho^2 - \eta y)^2 + \eta^2 x^2} - \frac{2x}{x^2 + (\eta - y)^2}\right]d\eta \qquad (7)\\ &+\sum_\nu \log\left[\frac{z - r_\nu e^{i\theta_\nu}}{\rho^2 - r_\nu e^{-i\theta_\nu} z}\cdot\frac{\rho_2 + r_\nu e^{i\theta_\nu} z}{z + r_\nu e^{-i\theta_\nu}}\right] = \log|f(z)|\end{aligned}$$

Before we proceed let us assume that there are no zeros inside a circle of radius $\leq |\tau|$. Let us first make $\rho \to \infty$ and then, making obvious limit transformations, we find that:

$$\begin{aligned}&-\frac{1}{2\pi}\int_{-1/2\tau}^{1/2\tau}\frac{4x\cos\theta\cos\phi\log|f(\zeta)|\,d\theta}{\rho}\\ &+\frac{1}{2\pi}\left\{\int_\tau^\rho + \int_{-\rho}^{-\tau} + \int_{-\tau}^{\tau}\right\}\log|f(i\eta)|\left[\frac{2\rho^2 x}{(\rho^2 - \eta y)^2 + y^2 x^2} - \frac{2x}{(\eta - x)^2 + x^2}\right]d\eta \qquad (8)\\ &+\sum_\nu \log\left[\frac{z - r_\nu e^{i\theta_\nu}}{(\rho^2 - r_\nu e^{i\theta_\nu} z)}\cdot\frac{\rho^2 + r_\nu e^{i\theta_\nu} z}{z + r_\nu e^{-i\theta_\nu}}\right] = \log|f(z)|.\end{aligned}$$

Next let us make $z$ approach zero along the real axis in such a way that $y = 0$ and $x \to 0$ and $\cos\phi = 1$.
To do this let us divide through by $r$ (or $x$) and pass to the limit as $r$ (or $x$) $\to 0$.
We write

$$\log \frac{(z - r_\nu e^{i\theta_\nu})\,(\rho^2 + r_\nu e^{i\theta_\nu} z)}{(\rho^2 - r_\nu e^{-i\theta_\nu} z)\,(z + r_\nu e^{-i\theta_\nu})} \tag{6}$$

as

$$\log\left(1 + r_\nu \frac{e^{i\theta_\nu} z}{\rho^2}\right) - \log\left(1 - \frac{r_\nu}{\rho^2} z e^{-i\theta_\nu}\right) + \log\left(\frac{z e^{-i\theta_\nu}}{r_\nu} - 1\right) - \log\left(\frac{z e^{i\theta_\nu}}{r_\nu} + 1\right). \tag{9}$$

This in turn may be expanded as a function

$$\log(1 + \epsilon) = \epsilon - \frac{\epsilon^2}{2} + \cdots, \qquad |\epsilon| < 1. \tag{10}$$

Neglecting terms of order greater than $\epsilon$, that is, in our case terms $O\left(\frac{1}{R^2}\right)$ we easily find the result which we now derive. However, we must still take care of the terms on the right hand side of our equation. To do this best we consider $\log f(z) - \int_{-\tau}^{\tau}$ as a single entity. We then find by an extremely simple calculation that

$$\lim_{z \to 0} \log f(z) - \int_{-\tau}^{\tau} \log |f(i\eta)| \left[\frac{2\rho^2 x}{(\rho^2 - \eta y)^2 + \eta^2 x^2} - \frac{2x}{x^2 + (y - \eta)^2}\right] d\eta \tag{11}$$

is finite where[3]

$$\log f(z) = \frac{1}{2\pi i} \int_D \log |f(\zeta)| \left\{\frac{\zeta + z}{\zeta - z} - \frac{\tau^2 - \zeta z}{\tau^2 + \zeta z}\right\} \frac{d\zeta}{\zeta}. \tag{12}$$

As a result we get:

$$-\frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \frac{\cos\theta \log |f(\zeta)|}{\rho}\, d\theta + \frac{1}{2\pi} \int_{-\rho}^{\rho} \log |f(i\eta)| \left(\frac{1}{\rho^2} - \frac{1}{\eta^2}\right) d\eta = -\frac{1}{2}\left[\sum_\nu \frac{r_\nu}{\rho^2}(e^{i\theta_\nu} + e^{-i\theta_\nu}) - \frac{1}{r_\nu}(e^{i\theta_\nu} + e^{-i\theta_\nu})\right] + O\left(\frac{1}{\rho^2}\right) + O(1). \tag{13}$$

[3] What we do essentially is to express $\log f(z)$ as an integral around the half-circle from $-\tau \to \tau$ and the short segment joining $-\tau \to \tau$.

In final form we get:

$$\sum_{\nu=1}^{n}\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\cos\theta_\nu=\frac{1}{\pi R}\int_{\pi/2}^{\pi/2}\log|f(Re^{i\theta})|\cos\theta\,d\theta$$
$$+\frac{1}{2\pi}\int_0^R\left(\frac{1}{y^2}-\frac{1}{R^2}\right)\log|f(iy)f(-iy)|\,dy+O(1). \tag{14}$$

This is the simplest case of the Carleman Theorem, which reads:

*Given the function $f(z)$ analytic for $x\geq 0$ with zeros $r_\nu e^{i\theta_\nu}$ then* (14) *holds.*

However, the methods used in this note could very easily be extended to give us results when we have poles and also to angular regions. As these results were found by other means originally and are well-known, I will only give indications as to the sort of results obtained. Again using the half-circle, we find, without difficulty, in the case where our functions have zeros and poles, that if $\gamma_\nu e^{i\phi_\nu}$ are the affixes of the poles, we get

$$\sum_\nu\log\frac{(z-r_\nu e^{i\theta_\nu})(\rho^2+r_\nu e^{i\theta_\nu}z)}{(\rho^2-r_\nu e^{-i\theta_\nu}z)(z+r_\nu e^{-i\theta_\nu})}$$
$$-\sum_\nu\log\frac{(z-\gamma_\nu e^{i\phi_\nu})(\rho^2+\gamma e^{i\phi_\nu}z)}{(\rho^2-\gamma_\nu e^{-i\phi_\nu}z)(z+\gamma_\nu e^{-i\phi_\nu})} \tag{15}$$

instead of (6). In the final result we get two terms:

$$\sum_\nu\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\cos\theta_\nu-\sum_\nu\left(\frac{1}{\gamma_\nu}-\frac{\gamma_\nu}{R^2}\right)\cos\phi_\nu. \tag{16}$$

If, following the usual method employed by Nevanlinna, we introduce the step functions:

for the zeros,

$$Z(\rho)=\sum_{r_\nu\leq\rho}r_\nu\cos\theta_\nu; \tag{17}$$

for the poles,

$$P(\rho)=\sum_{\gamma_\nu\leq\rho}\gamma_\nu\cos\phi_\nu; \tag{18}$$

it is immediately seen that

$$\sum_\nu\left(\frac{1}{r_\nu}-\frac{r_\nu}{R^2}\right)\cos\theta_\nu=z\int_0^R\frac{Z(r)\,dr}{r^2} \tag{19}$$

$$\sum_\nu\left(\frac{1}{\gamma_\nu}-\frac{\gamma_\nu}{R^2}\right)\cos\phi_\nu=z\int_0^R\frac{P(r)\,dr}{r^2} \tag{20}$$

Our result then reads:

$$\frac{1}{\pi R}\int_{-\pi/2}^{\pi/2}\log|f(Re^{i\theta})|\cos\theta\,d\theta + \frac{1}{2\pi}\int_0^R\left(\frac{1}{y^2}-\frac{1}{R^2}\right)[\log|f(iy)f(-iy)|]\,dy + O(1) = 2\left\{\int_0^R\frac{Z(r)-P(r)}{r^2}\,dr\right\}. \tag{21}$$

Furthermore, if, (following the procedure of Nevanlinna[4]) we let

$$n(r,0) = \sum_{r_\nu<r}\cos\theta_\nu, \tag{22}$$

$$n(r,\infty) = \sum_{r_\nu<r}\cos\phi_\nu \tag{23}$$

we easily find, using the Stieltjes integral,

$$\int_0^R[n(r,0)-n(r,\infty)]\left[\frac{1}{r^2}+\frac{1}{R^2}\right]dr = O(1) \tag{24}$$

+ same contour integral.

In closing it is very easy to give results analogous to Carleman's theorem in a sector by simply making a transformation $z=\zeta^k$. I will only give the result, as given by Nevanlinna: If in the sector $-\pi/2k \le \phi \le \pi/2k$ where $k \ge \frac{1}{2}$ we have a meromorphic function of $z=re^{i\phi}$ having as zeros

$$\alpha_n = |a_n|e^{ia_n}$$

and as poles (25)

$$\beta_n = |b_n|e^{i\beta_n}$$

and if we use the notation

$$n_k(r,0) = \sum_{|a_\nu|<r}\cos k\alpha_\nu, \qquad n_k(r,\infty) = \sum_{|b_\nu|<r}\cos k\beta_\nu \tag{26}$$

letting $n_k(r,0) - n_k(r,\infty) = n_k(r)$
then we get at once, where $S$ is the boundary of that part of the sector between the circles of radius $r_0$ and $R$

$$\int_{r_0}^R\left(\frac{1}{r^{k+1}}+\frac{r^{k-1}}{R^{2k}}\right)n_k(r)\,dr = \frac{1}{2\pi i}\int_S\log|f(z)|\left(\frac{1}{z^k}+\frac{z^k}{\rho^{2k}}\right)\frac{dz}{z}+O(1). \tag{27}$$

[4] R. Nevanlinna, Eindeutige Analytische Funktionen, 1936. Chaps. VI, VII, VIII.

Also, there are many known results concerning the order of magnitude of the function on the boundary and the frequency of zeros inside a contour of the above type. For example, Titchmarsh[5] proves that if $f(z)$ is analytic and bounded for $x \geq 0$ and has zeros in the right half-plane at $r_1 e^{i\theta_1}$, $r_2 e^{i\theta_2}$, $\cdots$ then the series

$$\sum_{n=1}^{\infty} \frac{\cos \theta_n}{r_n} < \infty \tag{28}$$

The method of proof used by Titchmarsh is, essentially, to say that

$$\sum_{r=1}^{n} \left(\frac{1}{r_r} - \frac{r_r}{R^2}\right) \cos \theta_r < M \quad \text{under the hypothesis given.} \tag{29}$$

Then it is possible, by taking $r_r < \frac{1}{2}R$ to prove that

$$\sum_{r_r < \frac{1}{2}R} \frac{\cos \theta_r}{r_r} < 4\,\frac{M}{3} \tag{30}$$

But letting $R \to \infty$ the result follows.

Using precisely the same argument for a function analytic and bounded in a sector, we can easily prove our result, using the fact that:

$$\begin{aligned} \int_{r_r}^{R} \frac{1}{r^{k+1}}\, n_k(r)\, dr &= -\int_{r_r}^{R} \frac{1}{k}\, d\left(\frac{1}{r^k}\right) n_k(r)\, dr \\ &= \frac{1}{k}\,\frac{1}{r^k}\, n_k(r) \int_{r}^{R} + \frac{1}{k} \int_{r}^{R} \frac{1}{r^k}\, dn_k(r). \end{aligned} \tag{31}$$

The first term is merely a constant; the second term behaves like

$$\frac{1}{k}\left(\sum_{1}^{\infty} \frac{\cos k\alpha_n}{|a_n|^k} - \frac{\cos k\beta_n}{|b_r|^k}\right).$$

For the second part of the integral we use a Titchmarsh argument. The integral on the right is bounded.

For a sector, the theorem now reads:

*If $f(z)$ is meromorphic for $x \geq 0$ and bounded, if we cut out the poles with small circles, and if its zeros are at $|\, a_r \,| e^{i\alpha_r}$ and its poles at $|\, b_r \,| e^{i\beta_r}$ then*

$$\sum_{n=1}^{\infty} \left(\frac{\cos k\alpha_n}{|a_r|^k} - \frac{\cos k\beta_r}{|b_r|^k}\right) < \infty. \tag{32}$$

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[5] E. C. Titchmarsh, Theory of Functions, 1932. P. 130 f.

# THE DISTRIBUTION OF VALUES OF AN ANALYTIC ALMOST PERIODIC FUNCTION IN EQUALLY SPACED CIRCLES

By R. H. Cameron

The distribution of values of almost periodic functions has been studied by[1] Bohr, Jessen, Wintner, and others; and of course the special case of the Riemann zeta function has a very extensive literature. However, these investigations deal with the average number of times a given value is assumed in an infinite strip, and the author is not aware of the existence of papers in which further localization of these values is discussed.

This paper deals with a system of equal circles equally spaced along a boundary line of an almost periodic strip; and shows that at least one in every so many of these circles must contain points where the function takes on specified values.

Lemma 1. *Let $f(z)$ be almost periodic in every strip interior to the strip $\alpha < R(z) < \beta$; and let $f(z)$ have the module $M$ and have a singularity at $z = \alpha$. Let $W_1, \cdots, W_\nu$ be any finite set of distinct complex numbers, and $Q$ any positive integer, and let*

$$\mathfrak{C}: \qquad \cdots C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$$

*be a sequence of equal circles whose centers are equally spaced on the line $R(z) = \alpha$ at the points $z = \alpha + (a + nl)i$. Let $\mathfrak{S}$ be the set of circles $C_n \in \mathfrak{C}$ such that either $f(z)$ has a singularity in $C_n$ or $f(z)$ takes on each of the values $W_1, \cdots, W_\nu$ in $C_n$ or $f(z)$ takes on all but one of the values $W_1, \cdots, W_\nu$ each at least $Q$ times in $C_n$. Then there exists a finite set of elements $\lambda_1', \cdots, \lambda_\mu'$ of $M$ and a number $\delta > 0$ such that $C_p \in \mathfrak{S}$ whenever $p$ satisfies*

$$|\lambda_k'(a + pl)| < \delta \pmod{2\pi} \qquad (k = 1, \cdots, \mu).$$

Suppose on the contrary that there is no such set as the $\lambda_1', \cdots, \lambda_\mu'$ and $\delta$ described. Then if $\lambda_1, \lambda_2, \cdots$ are the elements of $M$ arranged

[1] See, for instance, B. Jessen, "Ueber die Nullstellen einer analytischen fastperiodischen Funktionen. Eine Verallgemeinerung der Jensenschen Formel," Math. Ann. 108, 485–516 (1933); H. Bohr and B. Jessen, On the distribution of the values of the Riemann zeta function," Amer. J. of Math. 58, 35–44 (1936); B. Jessen and A. Wintner, "Distribution Functions and the Riemann Zeta Function," Trans. Amer. Math. Soc. 38, 48–88 (1935).

in some order, it follows that to each positive integer $n$ there corresponds a positive integer $p_n$ satisfying

$$(1) \qquad |\lambda_k(a + p_n l)| < \frac{1}{n} \pmod{2\pi} \qquad (k = 1, \cdots, n)$$

such that in $C_{p_n}$ the function $f(z)$ is regular and takes on not more than $\nu - 1$ of the values $W_1, \cdots, W_\nu$ and takes on not more than $\nu - 2$ of the values $W_1, \cdots, W_\nu$ as many as $Q$ times each. Since the sequence $p_1, p_2, \cdots$ is infinite and $W_1, \cdots, W_\nu$ is finite, it follows that the same $W_i$ is omitted infinitely often. Let $W$ be a member of the set $W_1, \cdots, W_\nu$ which is omitted infinitely often. Since the remaining $\nu - 1$ members of $W_1, \cdots, W_\nu$ are again a finite set, there is at least one of them, say $V \neq W$ which infinitely often occurs less than $Q$ times for values of $n$ for which $W$ is omitted. Let $p_1', p_2', \cdots$ be the subsequence of $p_1, p_2, \cdots$ such that in $C_{p_n'}$ the function $f(z)$ is regular and does not assume the value $W$ and does not assume the value $V$ more than $Q - 1$ times. Let $u$ be any fixed value of $(V - W)^{1/(2Q)}$.

Now if $r$ is the lesser of $\beta - \alpha$ and the radius of the circles $C_n$, and if $C$ is the circle of center $\alpha$ and radius $r$, define in $C$;

$$(2) \qquad g_n(z) = \frac{[f(z + (a + p_n' l)i) - W]^{\frac{1}{2Q}}}{2u} + \tfrac{1}{2};$$

where the $2Q$ valued function is to be made single valued by choosing a sheet on which we never have

$$\left\{[f(z + (a + p_n' l)i) - W]^{\frac{1}{2Q}}\right\}^2 = u^2 \qquad \text{in } C.$$

This can always be done for each $n$, for in the first place all $2Q$ sheets are separate and distinct in $C$ since the function is never equal to $W$; and in the second place if each of the $Q$ $Q$th roots becomes equal to $u^2$ in $C$, the function would equal $V$ $Q$ times, which is impossible. Thus the numerator of the main fraction in (2) never equals $\pm u$, and $g_n(z)$ never equals 1 or 0. We can therefore apply the Picard-Landau theorem to the functions $g_n(z)$ if we use a circle $C'$ interior to $C$, provided $g_n(z)$ is bounded in $n$ at the center of $C'$. Let us therefore take $C'$ as a circle of radius $r/3$ with center at the point $\alpha + (r/4)$; for the fact that $f(z)$ is almost periodic in $\alpha + \epsilon < R(z) < \beta - \epsilon$ implies that $g_n(\alpha + \frac{1}{4}r)$ is bounded in $n$. It follows that the functions $g_n(z)$ are uniformly bounded in $C'$.

Finally, we can choose a subsequence $h_1(z), h_2(z), \cdots$ of the functions

$g_1(z), g_2(z), \cdots$ which converges uniformly in every closed region interior to $C'$. Let $p_1'', p_2'', \cdots$ be the corresponding subsequence of $p_1', p_2', \cdots$, so that

$$h_n(z) = \frac{[f(z + (a + p_n'' l)i) - W]^{\frac{1}{2Q}}}{2u} + \tfrac{1}{2}; \tag{3}$$

and

$$\bar{h}(z) = \lim_{n \to \infty} h_n(z) \tag{4}$$

exists uniformly in every closed region interior to $C'$. Then from (1) we have

$$\lim_{n \to \infty} \lambda_k(a + p_n'' l) = 0 \pmod{2\pi}, \qquad (k = 1, 2, \cdots);$$

and since $f(z)$ is almost periodic,

$$\lim_{n \to \infty} f(z + (a + p_n'' l)i) = f(z)$$

uniformly in the strip $\alpha + \frac{1}{4}r < R(z) < \alpha + \frac{1}{2}r$. Hence

$$f(z) = [2u\bar{h}(z) - u]^{2Q} + W$$

in the right half of $C'$; and since the right member is analytic throughout $C'$, $f(z)$ has an analytic continuation throughout $C'$. But $C'$ contains the point $z = \alpha$ where by hypothesis $f(z)$ has a singularity; and this contradiction completes the proof of our Lemma.

From this Lemma and Kronecker's theorem on diophantine approximation we obtain immediately

THEOREM I. *Let $f(z)$ be almost periodic in every strip interior to the strip $\alpha < R(z) < \beta$; and let $f(z)$ have a singularity at $z = \alpha$. Let $W_1, \cdots, W_r$ be any finite set of distinct complex numbers, and $Q$ any positive integer, and let $\cdots C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ be a sequence of equal circles whose centers are equally spaced on the line $R(z) = \alpha$ at the points $z = \alpha + (a + nl)i$; the value of $a$ being 0 if a multiple of $2\pi/l$ belongs to the module $M$ of $f(z)$. Then there exists a positive integer $N$ with the following property:*

*If $C_{k+1}, C_{k+2}, \cdots, C_{k+N}$ are $N$ consecutive circles throughout the interiors of which $f(z)$ can be continued analytically without singular points, there is at least one of these circles in which $f(z)$ takes on all or all but one of the values $W_1, \cdots, W_r$; and if one of these values is actually omitted, the others are all taken on at least $Q$ times each.*

Applying this theorem to $\zeta(z)$, we obtain the

COROLLARY. *Let* $\cdots C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ *be a sequence of equal circles having their centers at the points* $\cdots 1 + (a - 2l)i$, $1 + (a - l)i$, $1 + ai$, $1 + (a + l)i$, $1 + (a + 2l)i$, $\cdots$ *and let* $W_1, \cdots, W_\nu$ *be any finite set of complex numbers and* $Q$ *any positive integer. Then there exists a positive integer* $N$ *such that among every* $N$ *consecutive circles* $C_{k+1}, C_{k+2}, \cdots, C_{k+N}$ *there is at least one circle* $C$ *in which the Riemann Zeta function takes on all of the values* $W_1, \cdots, W_\nu$ *with at most one exception; and furthermore if there is an exception, all the other* $\nu - 1$ *values are taken on at least* $Q$ *distinct times by the Zeta function. In particular, if the Riemann hypothesis is satisfied, any finite set of preassigned non-zero values will without exception be assumed at least* $Q$ *times each.*

All the statements of this corollary obviously follow from Theorem I if a multiple of $2\pi/l$ is not contained in the module of $\zeta(z)$ or if $a = 0$. However, due to the linear independence of the exponents in the product expansion of $\zeta(z)$ it is clear that $\pi/l$ will be independent of the exponents that remain after a single factor is removed. But this factor is a periodic function with no zeros in the strip considered, and Lemma 1 can be modified to cover the case where $f(z)$ is of the form $p(z) \cdot \bar{f}(z)$ in which $p(z)$ is periodic and $\bar{f}(z)$ has the module $M$.

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# INTERREFLECTIONS INSIDE AN INFINITE CYLINDER*

By William F. Whitmore

## Introduction

The first application of integral equations to the problem of interreflections between luminous surfaces seems to have been made by Yamauti,[1] though he did not state the integral equation itself, but merely derived the Fredholm solution for it. The equation was explicitly stated by Buckley[3] and a solution derived for the infinite cylinder by the use of a numerical approximation method. Perhaps the most extended discussion is that by Buckley,[4] in which the fundamental integral equation is derived and solutions are obtained for various forms of uniformly heated enclosures—such as the sphere and hemisphere, the infinite circular cylinder, the finite circular cylinder, and two semi-infinite parallel planes. The last three are handled by numerical approximation to the kernel of the integral equation by one and two term exponential solutions. In neither of the two papers here cited does Buckley consider the possibility of specifying separately the luminosity and reflection factor of the walls of the enclosure with which he is dealing. He is interested in suitable sources for black-body radiation and always expresses the reflection factor in terms of the emissivity of his enclosure—the latter he invariably supposes to be constant. This possibility is, however, considered by Yamauti,[2] who discusses the luminosity of certain types of translucent cylindrical sources, illuminated from within, such as are employed in architectural lighting. He com-

* This is in substance a thesis accepted as partial fulfillment of the requirement for the degree of Bachelor of Science by the Massachusetts Institute of Technology.

[1] Z. Yamauti: Light Flux Distribution of a System of Interreflecting Surfaces, Journal of the Optical Society of America, 13, (1926) p. 561.

[3] H. Buckley: On the Radiation from the Inside of a Circular Cylinder, Philosophical Magazine, iv, (1927), p. 753.

[4] H. Buckley: Some Problems of Interreflection, International Congress on Illumination, Proceedings, 1928, pp. 888–911.

[2] Z. Yamauti: Étude analytique des interréflexions dans un cylindre de longueur infinie, Revue Générale de l'Électricité, XLII #10, 4 Septembre 1937, pp. 293–299.

pares certain approximate solutions with exact solutions, referring to Buckley[4] for the derivation and statement of the latter.

In the present discussion I propose to state the integral equation for the infinite circular cylinder and show that it can be reduced to a second order differential equation of a somewhat simple type. I shall then show the solution for the cylinder with a longitudinal slit of different reflection factor from that of the remainder of the cylinder can be handled by using this equation in conjunction with the original integral equation. I shall finally give a brief sketch of a possible method of attack on the problem of infinite cylinders of arbitrary cross-section—a problem that does not seem to admit of any simple analytic solution. The particular novelty of this paper lies in the fact that simple analytic solutions are obtained for all the cases considered, in contrast to the rather inelegant numerical solutions previously given. In the derivations I have proceeded from the particular to the general, in the belief that this will make the discussion easier to comprehend than if the most general solution had been derived at the start and the particular solutions found as limiting cases of the general case.

### The Fundamental Integral Equation for Interreflections

I shall not attempt to derive the fundamental integral equation for interreflections. A very good discussion of the problem is to be found in Prof. Moon's recent book,[6] on which the terminology of this paper is based. For the case in which symmetry allows the position of a point to be expressed in terms of a single variable, the equation is:

$$L(x) = L_0(x) + \rho(x) \int_a^b L(y) \cdot K(y, x) \cdot dy \qquad (1)$$

where $L(x)$ is the actual luminosity at a point $x$ after interreflections have occurred,

$L_0(x)$ is the initial luminosity at $x$; i.e., the luminosity assuming no interreflections,

$\rho(x)$ is the reflection factor of the surface at $x$; in this discussion all surfaces are assumed to be perfectly diffusing, obeying the cosine law of emission,[7]

$L(y)$ is the actual luminosity at a point $y$,

$K(y, x)$ is the illumination at $x$ produced by unit luminosity of unit area at $y$, and is known, from physical considerations, to be symmetric.

[6] P. H. Moon: The Scientific Basis of Illuminating Engineering (McGraw-Hill, 1936), especially p. 327 and bibliography on p. 346.

[7] Ibid, p. 256.

Another relation which is needed, and which I shall also state without proof, is the illumination produced by a strip of infinite length and of width $ds$, which is at unit luminosity, at a point $P$ a distance $r$ from the strip. The angle between the direction of $r$ and the normal to the strip is denoted by $\theta$; that between $r$ and the normal to the surface at $P$ is $\phi$. The desired relation is:[8]

$$dE = \frac{\cos\theta\cos\phi}{2r}\,ds \qquad (2).$$

In the integral equation for the infinite cylinder of arbitrary cross-section, this factor $dE$ will replace $K(y, x)\cdot dy$. The use of this relation permits the reduction of the general integral equation for an infinite cylinder to one involving a contour integral around a cross-section of the cylinder. It allows the initial luminosity and the reflection factor to vary along the cross-section, but requires uniformity in both lengthwise of the cylinder. On the other hand, Buckley usually sets up a relation for the flux interchanged between cross-sectional elements and integrates along the length of the cylinder. This allows him to handle cylinders of finite length, but gives a kernel which probably cannot be handled analytically.

## The Complete Circular Cylinder

The figure represents the cross-section of an infinite circular cylinder. The point $P$ and the quantities $ds$, $r$, $\theta$, $\phi$ are those defined in the state-

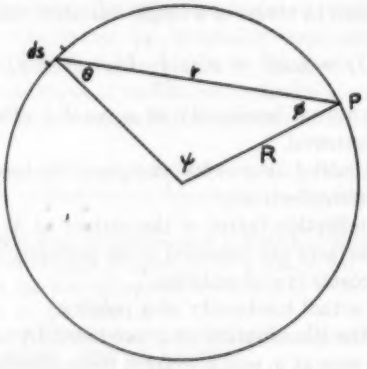


FIG. 1

ment of equation (2); $R$ is the radius of the cylinder and $\psi$ the central angle between $P$ and $ds$. Here $\theta = \phi = \frac{1}{2}(\pi - \psi)$; $\cos\theta\cos\phi = \sin^2\psi/2$;

[8] Ibid, p. 276.

$ds = Rd\psi$; $r^2 = 2R^2(1 - \cos\psi)$; $r = 2R \cdot \sin \frac{1}{2}\psi$. Hence equation (2) becomes:

$$dE = \tfrac{1}{4} \sin \tfrac{1}{2}\psi \cdot d\psi \tag{3}$$

and the integral equation for the infinite circular cylinder is:

$$L(x) = L_0(x) + \tfrac{1}{4}\rho(x) \int_0^{2\pi} L(y) \cdot \sin \tfrac{1}{2} |\, y - x \,| \cdot dy \tag{4}.$$

The absolute value signs on $y - x$ are to satisfy the physical requirement that the kernel of the equation should never become negative. They may be avoided by changing the range of integration so that $y - x$ is always positive, and equation (4) then becomes:

$$L(x) = L_0(x) + \tfrac{1}{4}\rho(x) \int_x^{x+2\pi} L(y) \cdot \sin \tfrac{1}{2}(y - x) \cdot dy \tag{5}.$$

A similar equation is given by Yamauti,[2] with the slight difference that it is expressed in terms of illumination and not luminosity. If $\rho$ is constant and it is noted that one must have $L(x) = L(x + 2\pi)$, the differential equation equivalent to (5) is easily obtained by differentiating twice with respect to $x$ and eliminating the integral term. (Primes denote derivatives with respect to $x$.)

$$L'(x) = L_0'(x) - \frac{\rho}{8} \int_x^{x+2\pi} L(y) \cdot \cos \tfrac{1}{2}(y - x) \cdot dy$$

$$L''(x) = L_0''(x) - \frac{\rho}{16} \int_x^{x+2\pi} L(y) \cdot \sin \tfrac{1}{2}(y - x) \cdot dy + \tfrac{1}{4}\rho L(x).$$

Multiplying this last result by 4, and adding it to equation (5), one obtains as the equivalent differential equation

$$4\, L''(x) + (1 - \rho) \cdot L(x) = L_0(x) + 4\, L_0''(x) \tag{6}.$$

After I had obtained this result, I discovered that Buckley[5] had stated a similar result, and it was undoubtedly to this that Yamauti referred for his exact solutions previously mentioned.

The integral equation (5) has now been reduced to the differential equation (6), under the supposition that $\rho$ is constant and that $L_0(x)$ has at least its first two derivatives. Since $\rho$ is by its very nature a positive (more exactly, non-negative) quantity and always less than unity, equation (6) is equivalent to the well-known equation for forced vibrations of a stretched string, with the periodic boundary conditions

[5] Ibid, in particular p. 902.

$L(x) = L(x + 2\pi)$. It has a general periodic solution, plus a particular integral depending on $\rho$ and $L_0(x)$. However, the general solution is not of period $2\pi$, so both arbitrary constants must vanish identically to satisfy the boundary conditions—the particular integral depends on functions of period $2\pi$, and hence will satisfy these conditions. The physical interpretation of the vanishing of the general solution in this case is that this solution satisfies the homogeneous equation with $L_0 = 0$, and its non-vanishing would give a positive luminosity resulting from zero initial luminosity—which is impossible.

In the very simplest case, with $L_0(x)$ a constant, the solution is $L(x) = \frac{1}{(1-\rho)} L_0$—identical with the solution of the same case for the sphere.[9] In connection with this observation, I should like to draw attention to the possibility of extending this particular solution to the case of any closed surface in three dimensions. This possibility may have been noted before, but I have not seen it stated in print, and I feel it is worthwhile to have an explicit statement and proof of the possibility. The proof depends on the "unit-sphere" principle,[10] which, briefly stated, says that the illumination produced at a given point by a perfectly diffusing surface of constant luminosity does not depend on the shape or curvature of the surface, but solely on the solid angle which it subtends at the point. Applying this principle to interreflections inside *any* closed surface in three dimensions, having a constant reflection factor $\rho$ and a constant initial luminosity $L_0$, it may be stated that the final luminosity $L$ will be given by $L = \frac{1}{(1-\rho)} L_0$. This result is perfectly general, and would apply, for example, to a cube, as well as to any continuously curved surface.

To return to the infinite circular cylinder with $\rho$ constant, the next simplest case is $L_0(x) = K \begin{matrix}\sin\\\cos\end{matrix} mx$. To solve equation (6) for this case, assume $L(x) = A \begin{matrix}\sin\\\cos\end{matrix} mx$. Equation (6) then becomes:

$$A(1 - 4m^2 - \rho) = K(1 - 4m^2).$$

This gives an actual luminosity for this case of

$$L(x) = K \frac{4m^2 - 1}{4m^2 - 1 + \rho} \begin{matrix}\sin\\\cos\end{matrix} mx \qquad (7).$$

[9] Ibid, p. 330.
[10] Ibid, p. 294.

For $m = 0$, this reduces to the solution for $L_0$ constant, as it should. The practical importance of this solution is that it holds for the typical Fourier series term, and thus permits handling any initial luminosity distribution which can be developed in a convergent Fourier expansion. Solutions for other types of initial distributions can be obtained by making such an expansion, or, in simple cases, by direct solution of equation (6).

It may not be amiss to verify the fact that the solution for $L_0(x) = K \cos mx$ actually satisfies the integral equation (5). This can be done by direct substitution, assuming $L(x) = A \cos mx$.

$$(A - K)\cdot\cos mx$$

$$= \tfrac{1}{4}\rho \int_x^{x+2\pi} A \cos my\cdot(\sin \tfrac{1}{2}y\cdot\cos \tfrac{1}{2}x - \cos \tfrac{1}{2}y\cdot\sin \tfrac{1}{2}x)\cdot dy$$

$$\int_x^{x+2\pi} \cos my\cdot\sin \tfrac{1}{2}(y - x)\cdot dy$$

$$= 2\left[\frac{\cos (m + \tfrac{1}{2})x\cdot\cos \tfrac{1}{2}x + \sin(m + \tfrac{1}{2})x\cdot\sin \tfrac{1}{2}x}{2m + 1}\right.$$

$$\left. - \frac{\cos(m - \tfrac{1}{2})x\cdot\cos \tfrac{1}{2}x - \sin(m - \tfrac{1}{2})x\cdot\sin \tfrac{1}{2}x}{2m - 1}\right]$$

$$= 2\left[\frac{\cos mx}{2m + 1} - \frac{\cos mx}{2m - 1}\right] = -\frac{4\cos mx}{4m^2 - 1}$$

$$K - A = \frac{\rho A}{4m^2 - 1}; \qquad A = \frac{4m^2 - 1}{4m^2 - 1 + \rho}K \qquad \text{Q.E.D.}$$

This verification, of course, includes a verification of the solution for $L_0(x)$ a constant ($m = 0$).

### Circular Cylinder with Variable $L_0$ and Variable $\rho$

For the sake of completeness, I shall now derive the equivalent differential equation for the infinite circular cylinder for the general case in which both $L_0(x)$ and $\rho(x)$ are allowed to vary. Starting again with equation (5):

$$L(x) = L_0(x) + \tfrac{1}{2}\rho(x) \int_x^{x+2\pi} L(y)\cdot\sin \tfrac{1}{2}(y - x)\cdot dy$$

and differentiating twice, one obtains successively

$$L'(x) = L_0'(x) - \frac{\rho(x)}{8}\int_x^{x+2\pi} L(y)\cdot\cos\tfrac{1}{2}(y-x)\cdot dy + \tfrac{1}{4}\rho'(x)\int_x^{x+2\pi} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy$$

$$L''(x) = L_0''(x) - \frac{\rho(x)}{16}\int_x^{x+2\pi} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy - \frac{\rho'(x)}{8}\int_x^{x+2\pi} L(y)\cdot\cos\tfrac{1}{2}(y-x)\cdot dy + \frac{\rho''(x)}{4}\int_x^{x+2\pi} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy + \tfrac{1}{4}\rho(x)\cdot L(x).$$

If the integral terms are eliminated between these three equations, the resulting differential equation is:

$$4L''(x) - 4\frac{\rho'(x)}{\rho(x)}L'(x) + \left[1 - \rho(x) + 4\frac{\rho'(x)^2}{\rho(x)^2} - 4\frac{\rho''(x)}{\rho(x)}\right]L(x) = L_0(x)\left[1 + 4\frac{\rho'(x)^2}{\rho(x)^2} - 4\frac{\rho''(x)}{\rho(x)}\right] - 4\frac{\rho'(x)}{\rho(x)}L_0'(x) + 4L_0''(x) \qquad (8).$$

Equation (8) is written in a form which shows at once that it reduces to equation (6) if $\rho$ is constant. It is a much less pleasant equation to deal with than equation (6); and so far as I am aware at present, it has very few practical applications. I am therefore confining myself to stating the equation, and shall not attempt to work out solutions for any particular forms of $\rho(x)$.

### Circular Cylinder with Longitudinal Slit on which $\rho = 0$

A case of more practical interest is that of the infinite cylinder with a longitudinal slit on which $\rho = 0$. Such a cylinder has been suggested as a source of black-body radiation;[5] and the problem also arises in calculating the luminosity of a cylindrical light source fastened to a longitudinal support. This discussion forms a useful preliminary to the case in which the reflection factor has one constant value over a portion of the cylinder and another constant value over the remainder of the cylinder. In the remaining part of this paper the reflection factor will be assumed to vary in this manner; i.e., it will be assumed to be piecewise constant.

The integral equation for the infinite circular cylinder with a longi-

tudinal slit is equation (4) with its limits changed to $\alpha$ and $\beta$; $\alpha \leq x \leq \beta$, $0 \leq (\beta - \alpha) \leq 2\pi$. To take advantage of the inherent symmetry of the problem, it is convenient to write $\omega = \frac{1}{2}(\beta - \alpha)$, and integrate from $-\omega$ to $+\omega$. With suitable regard for the requirement that the kernel be everywhere positive, the equation is:

$$L(x) = L_0(x) + \tfrac{1}{4}\rho \int_{-\omega}^{x} L(y) \cdot \sin \tfrac{1}{2}(x - y) \cdot dy + \tfrac{1}{4}\rho \int_{x}^{\omega} L(y) \cdot \sin \tfrac{1}{2}(y - x) \cdot dy \qquad (9).$$

Since extra terms in the differentiation arise only from variable limits, the differential equation equivalent to equation (9) is again equation (6), as can easily be verified. Indeed, the discussion of equation (6) could have been carried through from the start with general limits, but the boundary conditions for the complete cylinder are so simple and obvious that it seemed advisable to give a complete discussion of that case, avoiding any extra complications that might obscure the method. In this more general case, the differential equation seems to be useful chiefly in suggesting a suitable form for the solution, the arbitrary constants in which must be determined by substitution in the integral equation (9). It may be remarked in passing that equation (9) with variable reflection factor again reduces to equation (8), but that solutions of (8) are subject to the same necessity of checking in (9).

The general solution of equation (6) is:

$$L(x) = A \cdot \sin kx + B \cdot \cos kx + g(x) \qquad k^2 = \tfrac{1}{4}(1 - \rho) \qquad (10)$$

where $g(x)$ is the particular integral depending on $L_0(x)$ and is identical with the solutions already discussed for the case of the complete cylinder. The constants $A$ and $B$ are to be determined by substitution in the integral equation (9). In order to save space in carrying out this substitution, it is convenient to introduce a special notation for the integrals which will be required. (The symbol $\int^x$ denotes the indefinite integral of the quantity following it, with the arbitrary constant suitably chosen.)

We define:

$$F_{k1}(x) = \int^x \sin ky \cdot \sin \tfrac{1}{2}y \cdot dy = \frac{\sin (k - \frac{1}{2})x}{2k - 1} - \frac{\sin (k + \frac{1}{2})x}{2k + 1}$$

$$F_{k2}(x) = \int^x \sin ky \cdot \cos \tfrac{1}{2}y \cdot dy = -\frac{\cos (k - \frac{1}{2})x}{2k - 1} - \frac{\cos (k + \frac{1}{2})x}{2k + 1}$$

$$F_{k3}(x) = \int^x \cos ky \cdot \sin \tfrac{1}{2}y \cdot dy = \frac{\cos (k - \frac{1}{2})x}{2k-1} - \frac{\cos (k + \frac{1}{2})x}{2k+1}$$

$$F_{k4}(x) = \int^x \cos ky \cdot \cos \tfrac{1}{2}y \cdot dy = \frac{\sin (k - \frac{1}{2})x}{2k-1} + \frac{\sin (k + \frac{1}{2})x}{2k+1}$$

$$F_5(x) = \int^x g(y) \cdot \sin \tfrac{1}{2}y \cdot dy$$

$$F_6(x) = \int^x g(y) \cdot \cos \tfrac{1}{2}y \cdot dy$$

It will be observed that $F_{k1}$ and $F_{k4}$ are odd functions ($f(x) = -f(-x)$); $F_{k2}$ and $F_{k3}$ are even functions ($f(x) = f(-x)$). If one puts $k^2 = \frac{1}{4}(1 - \rho)$, as in equation (10), one may easily verify that

$$\tfrac{1}{2}A\rho\,[\sin \tfrac{1}{2}x \cdot F_{k2}(x) - \cos \tfrac{1}{2}x \cdot F_{k1}(x)] = \tfrac{1}{2}A\rho \left[\frac{1}{2k+1} - \frac{1}{2k-1}\right] \sin kx$$

$$= A \cdot \sin kx \qquad (11)$$

and similarly that

$$\tfrac{1}{2}B\rho\,[\sin \tfrac{1}{2}x \cdot F_{k4}(x) - \cos \tfrac{1}{2}x \cdot F_{k3}(x)] = B \cdot \cos kx. \qquad (12)$$

Furthermore, the assumption that

$$\tfrac{1}{2}\rho\,[\sin \tfrac{1}{2}x \cdot F_6(x) - \cos \tfrac{1}{2}x \cdot F_5(x)] = g(x) - L_0(x) \qquad (13)$$

yields, on differentiating both sides twice with respect to $x$ (and taking account of the terms from the variable upper limit):

$$g'(x) - L_0'(x) = \tfrac{1}{4}\rho\,[\cos \tfrac{1}{2}x \cdot F_6(x) - \sin \tfrac{1}{2}x \cdot F_5(x)]$$

$$g''(x) - L_0''(x) = -\rho/8\,[\sin \tfrac{1}{2}x \cdot F_6(x) - \cos \tfrac{1}{2}x \cdot F_5(x) - 2g(x)]$$

$$= \tfrac{1}{4}[L_0(x) - (1 - \rho)g(x)]$$

which last relation is identical with the differential equation by which $g(x)$ was determined from $L_0(x)$, and therefore justifies our assumption. As will be evident in a moment, the effect of these three relations (11, 12, 13) is to show that the terms in $x$ on the righthand side of the integral equation (9) will cancel out those on the left when one substitutes

$$L(x) = A \cdot \sin kx + B \cdot \cos kx + g(x).$$

Making this substitution and writing out equation (9) in full in terms of the $F$-functions, one obtains:

$$
\begin{aligned}
A \cdot \sin kx + B \cdot \cos kx + g(x) - L_0(x) &= \tfrac{1}{4}A\rho \sin \tfrac{1}{2}x\,[2F_{k2}(x) - F_{k2}(\omega) - F_{k2}(-\omega)] \\
&- \tfrac{1}{4}A\rho \cos \tfrac{1}{2}x\,[2F_{k1}(x) - F_{k1}(\omega) - F_{k1}(-\omega)] \\
&+ \tfrac{1}{4}B\rho \sin \tfrac{1}{2}x\,[2F_{k4}(x) - F_{k4}(\omega) - F_{k4}(-\omega)] \\
&- \tfrac{1}{4}B\rho \cos \tfrac{1}{2}x\,[2F_{k3}(x) - F_{k3}(\omega) - F_{k3}(-\omega)] \\
&+ \tfrac{1}{4}\rho \sin \tfrac{1}{2}x\,[2F_5(x) - F_5(\omega) - F_5(-\omega)] \\
&- \tfrac{1}{4}\rho \cos \tfrac{1}{2}x\,[2F_6(x) - F_6(\omega) - F_6(-\omega)].
\end{aligned}
$$

By reason of equations (11, 12, 13), the terms in $x$ on the right of this equation cancel all the terms on the left; also the fact that $F_{k1}$ and $F_{k4}$ are odd can be used to eliminate four more terms, and the fact that $F_{k2}$ and $F_{k3}$ are even can be used to reduce four other terms to two terms. Employing all these simplifications, one obtains:

$$
\begin{aligned}
0 = {} & [2BF_{k3}(\omega) + F_5(\omega) + F_5(-\omega)] \cos \tfrac{1}{2}x \\
& - [2AF_{k2}(\omega) + F_6(\omega) + F_6(-\omega)] \sin \tfrac{1}{2}x \qquad (14)
\end{aligned}
$$

as a relation which must be satisfied identically in order to have relation (10) satisfy the integral equation (9). Since the sine and cosine are linearly independent, the only way in which (14) can be satisfied identically is to have the coefficients vanish separately. We now have $A$ and $B$ defined in terms of known functions and their integrals, and all that remains is to evaluate $F_5$ and $F_6$ in terms of the correct $g(x)$ for the given $L_0(x)$.

In particular, suppose that $L_0(x) = L_s \sin mx + L_c \cos mx$, then

$$
g(x) = \frac{4m^2 - 1}{4m^2 - 1 + \rho}\,(L_s \sin mx + L_c \cos mx) \text{ and}
$$

$$
F_5(x) = \frac{4m^2 - 1}{4m^2 - 1 + \rho}\,[L_s F_{m1}(x) + L_c F_{m3}(x)]
$$

$$
F_6(x) = \frac{4m^2 - 1}{4m^2 - 1 + \rho}\,[L_s F_{m2}(x) + L_c F_{m4}(x)].
$$

These are, of course, the same $F$-functions as before, with $m$ replacing $k$ as the parameter. Substituting these relations in (14), using the

even and odd properties of the $F$-functions, and equating the coefficients to zero, one obtains:

$$A = -\frac{4m^2-1}{4m^2-1+\rho}\frac{F_{m2}(\omega)}{F_{k2}(\omega)}L_s$$

$$= \frac{\rho L_s}{4m^2-1+\rho}\frac{2m\cdot\cos m\omega\cdot\cos\tfrac{1}{2}\omega+\sin m\omega\cdot\sin\tfrac{1}{2}\omega}{2k\cdot\cos k\omega\cdot\cos\tfrac{1}{2}\omega+\sin k\omega\cdot\sin\tfrac{1}{2}\omega} \tag{15}$$

$$B = -\frac{4m^2-1}{4m^2-1+\rho}\frac{F_{m3}(\omega)}{F_{k3}(\omega)}L_c$$

$$= \frac{\rho L_c}{4m^2-1+\rho}\frac{\cos m\omega\cdot\cos\tfrac{1}{2}\omega+2m\cdot\sin m\omega\cdot\sin\tfrac{1}{2}\omega}{\cos k\omega\cdot\cos\tfrac{1}{2}\omega+2k\cdot\sin k\omega\cdot\sin\tfrac{1}{2}\omega} \tag{16}$$

as the explicit form of the constants $A$ and $B$ for an initial luminosity distribution which is a typical Fourier series term. Since $m$ is always taken to be an integer, these constants both vanish identically for $\omega = \pi$; and the solution reduces to $g(x)$, the solution for the complete cylinder. The solution for constant initial luminosity is obtained by putting $m = 0$, in which case $A$ vanishes identically. An interesting feature of these equations is that a sine or cosine term in the actual luminosity arises only if a sine or cosine term is present in the initial luminosity, verifying the natural expectation that a symmetric original distribution should lead to a symmetric actual distribution. For the slit ($x > \omega$), both the reflection factor and initial luminosity are zero; therefore the actual luminosity of the slit is zero, as it should be. Incidentally, those acquainted with the theory of differential equations will realize that the extreme simplicity of the solutions for $\sin mx$ or $\cos mx$ is due to the fact that these are the proper eigenfunctions for the differential equation (6).

To recapitulate: For an infinite circular cylinder of reflection factor $\rho$, with a longitudinal slit of width $2\omega$ on which the reflection factor is zero, and having an initial luminosity given by $L_0 = L_s \sin mx + L_c \cos mx$, the final luminosity is

$$L(x) = A\cdot\sin kx + B\cdot\cos kx + \frac{4m^2-1}{4m^2-1+\rho}(L_s\sin mx + L_c\cos mx) \tag{17}$$

where $k^2 = \frac{1}{4}(1-\rho)$ and $A$ and $B$ are determined by equations (15) and (16).

### Circular Cylinder with piecewise constant $\rho$

Suppose that one is given an infinite circular cylinder with $\rho = \rho_1$ over $-\omega \leq x \leq \omega$, which will be referred to as interval (a), and that

$\rho = \rho_2$ over $\omega \leq x \leq 2\pi - \omega$ (interval (b)). Equation (9) is now replaced by two integral equations, one for $x$ in (a) and the other for $x$ in (b).

$$L(x) = L_0(x) + \frac{\rho_1}{4}\left[\int_{-\omega}^{x} L(y)\cdot\sin\tfrac{1}{2}(x-y)\cdot dy + \int_{x}^{\omega} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy\right] + \frac{\rho_2}{4}\int_{\omega}^{2\pi-\omega} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy \tag{18}$$

$$L(x) = L_0(x) + \frac{\rho_1}{4}\int_{-\omega}^{\omega} L(y)\cdot\sin\tfrac{1}{2}(x-y)\cdot dy + \frac{\rho_2}{4}\left[\int_{\omega}^{x} L(y)\cdot\sin\tfrac{1}{2}(x-y)\cdot dy + \int_{x}^{2\pi-\omega} L(y)\cdot\sin\tfrac{1}{2}(y-x)\cdot dy\right] \tag{19}.$$

Equation (18) holds for interval (a); equation (19) for interval (b). Naturally, there are also two different solutions for $L(x)$ corresponding to these two intervals:

(a) $$L(x) = A\cdot\sin kx + B\cdot\cos kx + g_1(x) \qquad k^2 = \tfrac{1}{4}(1-\rho_1) \tag{20}$$

(b) $$L(x) = C\cdot\sin hx + D\cdot\cos hx + g_2(x) \qquad h^2 = \tfrac{1}{4}(1-\rho_2) \tag{21}$$

Incidentally, it is obvious that if $\rho_2 = 0$, equation (19) drops out, as does (21); and (18) and (20) reduce to the equation and solution considered in the preceding section. It is also evident that if the correct $L(x)$ for the interval over which one is integrating in (18) and (19) is substituted in these equations, the terms in $x$ on the right again cancel those on the left, and one is left with two equations similar to (14), but with added terms from the extra integral in each equation. We consider as before $L_0(x) = L_s \sin mx + L_c \cos mx$ and obtain $g_1(x) = \gamma_1 L_0(x)$ with $\gamma_1 = \dfrac{4m^2-1}{4m^2-1+\rho_1}$; $g_2(x) = \gamma_2 L_0(x)$ with $\gamma_2 = \dfrac{4m^2-1}{4m^2-1+\rho_2}$; also let $F_{1s}$ be the $F_s$ for $g_1$, and similarly for the functions $F_{1c}$, $F_{2s}$, $F_{2c}$.

By an obvious extension of the methods of the preceding section we obtain two algebraic equations from (18) and (19).

$$0 = 2\left[BF_{k3}(\omega) + \gamma_1 L_c F_{m3}(\omega)\right]\cos\tfrac{1}{2}x - 2\left[AF_{k3}(\omega) + \gamma_1 L_s F_{m3}(\omega)\right]\sin\tfrac{1}{2}x$$

$$+\frac{\rho_2}{\rho_1}C\,[F_{A1}(2\pi-\omega)-F_{A1}(\omega)]\cos\tfrac{1}{2}x$$

$$-\frac{\rho_2}{\rho_1}C\,[F_{A2}(2\pi-\omega)-F_{A2}(\omega)]\sin\tfrac{1}{2}x$$

$$+\frac{\rho_2}{\rho_1}D\,[F_{A3}(2\pi-\omega)-F_{A3}(\omega)]\cos\tfrac{1}{2}x$$

$$-\frac{\rho_2}{\rho_1}D\,[F_{A4}(2\pi-\omega)-F_{A4}(\omega)]\sin\tfrac{1}{2}x$$

$$+\frac{\rho_2}{\rho_1}[F_{25}(2\pi-\omega)-F_{25}(\omega)]\cos\tfrac{1}{2}x$$

$$-\frac{\rho_2}{\rho_1}[F_{26}(2\pi-\omega)-F_{26}(\omega)]\sin\tfrac{1}{2}x$$

$$0=\frac{\rho_2}{\rho_1}C[F_{A1}(2\pi-\omega)+F_{A1}(\omega)]\cos\tfrac{1}{2}x$$

$$-\frac{\rho_2}{\rho_1}C[F_{A2}(2\pi-\omega)+F_{A2}(\omega)]\sin\tfrac{1}{2}x$$

$$+\frac{\rho_2}{\rho_1}D[F_{A3}(2\pi-\omega)+F_{A3}(\omega)]\cos\tfrac{1}{2}x$$

$$-\frac{\rho_2}{\rho_1}D[F_{A4}(2\pi-\omega)+F_{A4}(\omega)]\sin\tfrac{1}{2}x$$

$$+\frac{\rho_2}{\rho_1}[F_{25}(2\pi-\omega)+F_{25}(\omega)]\cos\tfrac{1}{2}x-\frac{\rho_2}{\rho_1}[F_{26}(2\pi-\omega)+F_{26}(\omega)]\sin\tfrac{1}{2}x$$

$$-2[AF_{A1}(\omega)+\gamma_1L_eF_{m1}(\omega)]\cos\tfrac{1}{2}x+2[BF_{A4}(\omega)+\gamma_1L_eF_{m4}(\omega)]\sin\tfrac{1}{2}x$$

Using the explicit expressions for $F_{25}$ and $F_{26}$ in terms of $F_m$-functions (cf. bottom of p. 10) and the fact that if $m$ is an integer $F_{m1}(2\pi-\omega)=+F_{m1}(\omega)$; $F_{m2}(2\pi-\omega)=-F_{m2}(\omega)$; $F_{m3}(2\pi-\omega)=-F_{m3}(\omega)$; $F_{m4}(2\pi-\omega)=+F_{m4}(\omega)$ (if the parameter is not an integer, there exists no simple relation between these functions), one obtains four simultaneous equations for the four coefficients $A$, $B$, $C$, $D$ on equating the coefficients of $\cos\frac{1}{2}x$ and $\sin\frac{1}{2}x$ to zero in each of the two equations.

These four equations are:

$$\rho_2C[F_{A1}(2\pi-\omega)+F_{A1}(\omega)]+\rho_2D[F_{A3}(2\pi-\omega)+F_{A3}(\omega)]$$
$$-2\rho_1AF_{A1}(\omega)=2(\rho_1\gamma_1-\rho_2\gamma_2)L_eF_{m1}(\omega)$$

$$\rho_2 C[F_{s1}(2\pi - \omega) - F_{s1}(\omega) + \rho_2 D[F_{s2}(2\pi - \omega) - F_{s2}(\omega)] + 2\rho_1 B F_{s2}(\omega) = 2(\rho_2\gamma_2 - \rho_1\gamma_1)L_e F_{m3}(\omega)$$

$$\rho_2 C[F_{s2}(2\pi - \omega) + F_{s2}(\omega)] + \rho_2 D[F_{s4}(2\pi - \omega) + F_{s4}(\omega)] - 2\rho_1 B F_{s4}(\omega) = 2(\rho_1\gamma_1 - \rho_2\gamma_2)L_e F_{m4}(\omega)$$

$$\rho_2 C[F_{s2}(2\pi - \omega) - F_{s2}(\omega)] + \rho_2 D[F_{s4}(2\pi - \omega) - F_{s4}(\omega)] + 2\rho_1 A F_{s2}(\omega) = 2(\rho_2\gamma_2 - \rho_1\gamma_1)L_e F_{m2}(\omega)$$

which can be solved uniquely for the coefficients, since the matrix of the coefficients of $A$, $B$, $C$, $D$ does not vanish. The actual details of the solution are somewhat tedious and the explicit expressions for $A$, $B$, $C$, $D$ are of no particular theoretical interest, so I shall leave them in the implicit form of the four simultaneous equations. This method can, of course, be immediately extended to the case of $n$ slits with different reflection factors, providing one is willing to set up and solve the corresponding $2n$ simultaneous linear equations for the $2n$ coefficients of the expressions for the actual luminosity.

### Remarks on the Cylinder of Arbitrary Cross-section

My first idea on considering the cylinder of arbitrary cross-section was to try to apply the theory of conformal mapping by means of complex functions, in combination with some modification of the unit-sphere principle.[10] However, after several fruitless attempts, I was forced to conclude that the problem was not of such a nature as to permit this; and Profs. Philip Franklin and W. T. Martin of the M. I. T. Department of Mathematics, with whom I discussed the matter, agreed with this conclusion. It may be that the reasons for this situation are analogous to those which make the application of vector analysis to illumination so unsatisfactory; if so, its final solution will probably have to await the formulation of a satisfactory theory of the illumination field.

Another approach would be to express the cross-section in terms of its intrinsic or natural variables—arc length and curvature[11]—, take arc length as the independent variable, and express the quantities in equation (2) in terms of a double Fourier series in $s_1$ and $s_2$ taken as any two points on the cross-section (since the actual length of the curve is immaterial, it may be taken as $2\pi$). One could then take the initial and

[11] W. C. Graustein: Differential Geometry (Macmillan, 1935) or any similar work.

actual luminosity as single Fourier series in $s_1$ and $s_2$ respectively and approximate to the actual values by a suitable number of terms. The orthogonality properties of sin $mx$ and cos $mx$ could be used to reduce the integral equation to a set of linear simultaneous equations for the coefficients of the terms approximating to the actual luminosity in terms of the approximating terms for the initial luminosity and the kernel.

Finally, one could return to Buckley's method of numerical approximation to the kernel by means of exponential terms. Either of these last two methods would require a fair amount of computing labor and cannot be counted on to furnish very general results; but I cannot see much hope of getting analytic results for any cases of much greater generality than those I have here considered, until further results are available from the theory of integral equations.

**Acknowledgments**

I should like to express my appreciation to Prof. Parry H. Moon of the M. I. T. Electrical Engineering Department, who first called this problem to my attention and has been always ready with advice on the illumination aspect of it, and to Prof. Philip Franklin of the Department of Mathematics, who suggested a suitable method of attacking the mathematical problems.

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# ON SINGULAR DISTRIBUTIONS

By Norbert Wiener and Aurel Wintner

Let $0 < a < 1$. Then $L_a(u) = \prod_{n=1}^{\infty} \cos(a^n u)$ is convergent for $-\infty < u < +\infty$. It is known that $L_a(u)$ is the Fourier-Stieltjes transform of a symmetric Bernoulli convolution $\sigma_a = \sigma_a(x)$, $-\infty < x < +\infty$; that the distribution function $\sigma_a(x)$ has no discontinuities and is constant for $|x| > a/(1-a)$, but not for $|x| > -\epsilon + a/(1-a)$; that $\sigma_a(x)$ is either nowhere or almost everywhere constant on the interval $|x| \leq a/(1-a)$ according as $a \geq \frac{1}{2}$ or $a < \frac{1}{2}$. It is not known whether $\sigma_a(x)$ is absolutely continuous for every $a \geq \frac{1}{2}$. All that is clear is that $\sigma_a(x)$ is absolutely continuous for $a = 2^{-1/\nu}$ ($\nu = 1, 2, \cdots$), and that it acquires derivatives of arbitrarily high order as $\nu \to +\infty$. This follows from the multiplication rule of Fourier-Stieltjes transforms, since $L_{\frac{1}{2}}(u)$ is Vieta's product for $\sin u/u$, which is $O(|u|^{-1})$ as $u \to \pm\infty$.

Thus $\sigma_{\frac{1}{2}}(x)$ is linear ($= \frac{1}{2}x + \frac{1}{2}$) on its range $|x| \leq 1 = a/(1-a)$. Let $a < \frac{1}{2}$. Then[1] $\sigma_a(x)$ is of the Cantor type, $\sigma_a(x)$ being constant on a dense set of subintervals of $|x| \leq a/(1-a)$. In particular, $\sigma_{\frac{1}{2}}(x)$ is the classical Cantor function on $|x| \leq 2$. Kershner[2] has shown that, while the Fourier-Stieltjes transform $L_a(u)$ does not tend to 0 if $a = 1/q$, where $q = 3, 4, 5, \cdots$, we have $L_a(u) = O([\log|u|]^{-b})$, if $a$ is a rational number $p/q$, not the reciprocal of an integer, while $b$ is a positive function of $p$ and $q$. Nothing is known about $L_a(u)$ if $a$ is irrational. On the other hand, as $a \to \frac{1}{2} - 0$, the singular function $\sigma_a(x)$ tends to the linear function $\sigma_{\frac{1}{2}}(x)$.

We shall show in §1 that, no matter what the arithmetical structure of $a$ may be, we shall have for every fixed $a < \frac{1}{2}$

$$\frac{1}{2u}\int_{-u}^{u} (L_a(v))^2\,dv = O\left(|u|^{\frac{\log 2}{\log a}}\right) \qquad (u \to \infty).$$

Thus the possible irregularities are smoothed out by averaging with respect to $u$.

Upon choosing $a$ sufficiently near to $\frac{1}{2}$, it follows that there exists for

[1] R. Kershner and A. Wintner, Amer. Journ. Math. 57 (1935), 541–548.

[2] R. Kershner, Amer. Journ. Math. 58 (1936), 450–452.

every $\epsilon > 0$ a continuous distribution function which is constant almost everywhere and has a Fourier-Stieltjes transform which is $O(|u|^{\epsilon-1})$ on the average. Combining this with a device applied recently by the authors,[3] it follows that there exists for every $\epsilon > 0$ a monotone function $f(x)$, defined over the range $-\pi \leq x \leq \pi$, of the Cantor type, and with Fourier-Stieltjes constants $c_n = \bar{c}_{-n}$ such that $c_n = O(n^{-\frac{1}{2}+\epsilon})$. The functions of the previous paper, while singular, are not of the Cantor type, since they are not almost everywhere constant.

In §2, we shall apply the method of §1 to the weakly convergent products $\prod(1 + \cos q_n x)$, instead of to the Fourier-Stieltjes transforms $\prod \cos(a^n u)$. We shall suppose for the sake of simplicity that the $q_n$ are positive integers. It is known[4] that if $q_{n+1}/q_n > \lambda$ holds for a fixed $\lambda > 3$,

$$\varphi_m(x) = -x + \int_0^x \prod_{n=1}^{m} (1 + \cos q_n y)\, dy$$

tends as $m \to \infty$ to a singular function $\varphi(x) = \varphi(x + 2\pi)$. We shall show that in order to assure the existence of a limit function $\varphi(x)$ which shall be continuous, of bounded variation, but not absolutely continuous, we may replace $\lambda > 3$ by $\lambda > 2$.

Corresponding to the fact that $L_a(u)$ becomes absolutely continuous for $a = \frac{1}{3}$, it is to be expected that the result $\lambda > 2$ is the best possible of its kind, so that $\lambda = 2$ is not admissible. This is actually the case. In fact, we shall verify in §3 that if $q_n = 2^n$, $\varphi(x)$ exists but is discontinuous.

In §4, we shall point out a parallelism between the results of §§2 and 3 and a celebrated Gegenbeispiel of Bohr.[5] Bohr's Gegenbeispiel is defined by a recursion formula tantamount to an infinite convolution, and hence its Fourier transform is defined by an infinite product, of which we can give the factors explicitly. This will be a weakly convergent infinite product not unlike those which we have discussed in the last two paragraphs.

§5 is devoted to the filling in of a lacuna in Wiener's general theory of spectral functions.[6] According to this theory, every autocorrelation of a function possessed of a spectrum, upon suitable choice of units,

[3] N. Wiener and A. Wintner, Amer. Journ. Math. 60 (1938), 513–532.

[4] Cf. p. 130 and p. 294 of A. Zygmund's Trigonometrical Series (1935), where further references are given.

[5] H. Bohr, Acta Math. 45 (1925), 112–117.

[6] N. Wiener, Acta Math. 55 (1931), 117–258, Chap. II.

is the Fourier-Stieltjes transform of a distribution function. It is quite obvious that if a distribution function is absolutely continuous, or if it is purely discontinuous, it may be obtained as the function of which some autocorrelation is the Fourier-Stieltjes transform. Up to the present, the only published example of an autocorrelation with singular Fourier-Stieltjes transform is one given by Mahler.[7] We shall show in this paragraph that the problem has nothing to do with the structure of particular singular distribution functions, since *every* given distribution function belongs to the spectrum analysis of the autocorrelation of a suitable chosen function. In this sense, the result of §5 is that Wiener's theory of spectrum analysis is identical with the theory of the Fourier-Stieltjes transforms of distribution functions. The proof will be based on a diagonal process which is made possible by the continuity theorem of P. Lévy.[8]

Finally, while the result of §5 makes the exhibition of specific singular spectra superfluous from the point of view of existence proofs, we shall nevertheless devote §6 to the construction of one special example, with a theory simpler than that of Mahler, and closely allied to the theory of infinite convolutions.

§1. Let $0 < a < \frac{1}{2}$ and $u \geq 0$. Then if $L_a(u) = \prod_{k=1}^{\infty} \cos(a^k u)$, we have for every $n$,

$$\int_0^u [L_a(v)]^2\,dv \leq \int_0^u \left[\prod_{k=1}^{n} \cos(a^k v)\right]^2 dv = 2^{-2n} \int_0^u \left|\prod_{k=1}^{n} (e^{ia^k v} + e^{-ia^k v})\right|^2 dv.$$

Hence

$$\int_0^u [L_a(v)]^2\,dv \leq 2^{-2n} \int_0^u |\sum \exp\{i(\pm a \pm a^2 \pm \cdots \pm a^n)v\}|^2\,dv.$$

Now write the integrand $|\sum|^2 = \sum\sum$ as a linear combination of terms $e^{ibv}$, and count the number of terms which belong to the same frequency $b$. Let us notice that $\sum_{k+1}^{\infty} a^j = \frac{a^{k+1}}{1-a} = a^k \frac{a}{1-a} > a^k$. On

[7] K. Mahler, Journ. Math. Phys. Mass. Inst. Technology 6 (1927), 158–164.
[8] P. Lévy, C. R. 175 (1922), 854–856.

integrating between $v = 0$ and $v = u$, we obtain from $e^{ibv}$ the divisor $b$ or the factor $u$ according as $b \neq 0$ or $b = 0$. Thus it is clear that

$$\int_0^u [L_a(v)]^2\,dv < \frac{\text{const.}}{2^{2n}}\left(2^n u + \sum_{k=0}^{n} \frac{2^{n+k}}{a^{n-k}}\right).$$

This may be written as

$$\int_0^u [L_a(v)]^2\,dv < \text{const.}\left(\frac{u}{2^n} + \frac{1}{(2a)^n}\sum_{k=0}^{n}(2a)^k\right).$$

Now let us again make use of the assumption that $a < \frac{1}{2}$. We have $\sum_{k=0}^{\infty}(2a)^k < +\infty$, and thus

$$\int_0^u [L_a(v)]^2\,dv < \text{const.}\left(\frac{u}{2^n} + \frac{1}{(2a)^n}\right)$$

for every $u \geq 1$ and $n \geq 1$. Choose $n = \left[\frac{\log u}{\log 1/a}\right]$ for every $u$. Then if we let $u \to \infty$, it follows that

$$\int_0^u [L_a(v)]^2\,dv = O\left(u^{\frac{\log 2a}{\log a}}\right) + O\left(u^{\frac{\log 2a}{\log a}}\right)$$

for every fixed $a < \frac{1}{2}$. However, $L_a(u) = L_a(-u)$, so that

$$\frac{1}{2u}\int_{-u}^{u} [L_a(v)]^2\,dv = O\left(u^{\frac{-\log 2}{\log 1/a}}\right),$$

as we have stated in the introduction.

Apparently $O$ cannot be replaced by $o$. Actually $\dfrac{\log 2}{\log 1/a}$ is precisely[9] the Lipschitz exponent of the distribution function $\sigma_a(x)$ whose Fourier-Stieltjes transform is $L_a(u)$.

§2. Let $q_1, \cdots, q_n, \cdots$ be a sequence of positive integers. Put

$$\varphi_n(x) = -x + \int_0^x \prod_{k=1}^{n}(1 + \cos q_k y)\,dy,$$

so that $\varphi_n(x)$ is a trigonometrical polynomial and $\varphi_n(x) + x$ a strictly increasing function for $-\infty < x < \infty$. Clearly,

$$\varphi_{n+1}(x) - \varphi_n(x) = \int_0^x (\cos q_{n+1} y) \prod_{k=1}^{n}(1 + \cos q_k y)\,dy.$$

[9] F. Hausdorff, Math. Annalen 79 (1918), 157–179; cf. R. Kershner and A. Wintner, loc. cit. 1.

Now suppose that $q_{n+1}/q_n \geq \lambda$ for some fixed $\lambda > 2$. Then

$$\prod_{k=1}^{n} (1 + \cos q_k y) = \sum_{j=1}^{m} p_j \cos r_j y,$$

where $m$ and $r_j$ are non-negative integers and the $p_j$ are non-negative numbers which depend on $n$ in such a way that

$$\sum_{j=1}^{m} p_j = 2^n \quad \text{and} \quad r_j \leq \sum_{k=1}^{n} q_k \leq q_{n+1} \frac{1}{\lambda - 1},$$

$1/(\lambda - 1)$ being obtained by using the geometrical progression as majorant. Thus,

$$|\varphi_{n+1}(x) - \varphi_n(x)| < \text{const.} \sum_{j=1}^{m} \frac{p_j}{q_{n+1} - r_j},$$

where const. is independent of $x$ and $n$. Furthermore, it is clear that

$$\sum_{j=1}^{m} \frac{p_j}{q_{n+1} - r_j} < \text{const} \sum_{n=1}^{m} \frac{p_n}{q_{n+1}} < \text{const} \left(\frac{2}{\lambda}\right)^n.$$

It follows, therefore, from $\lambda > 2$ that $\sum |\varphi_{n+1}(x) - \varphi_n(x)|$ is majorized by a fixed convergent geometrical progression; so that, as $n \to \infty$, $\varphi_n(x)$ tends uniformly to a limit function $\varphi(x)$.

This continuous $\varphi(x)$ has the period $2\pi$ and is of bounded variation. In fact, since every $\varphi_n(x) + x$ is continuous and monotone, while every $\varphi_n(x)$ has the period $2\pi$, these properties carry over in the limit. Finally, $\varphi(x)$ cannot be absolutely continuous over $-\pi \leq x \leq \pi$. If it were, by the Riemann-Lebesgue lemma, its Fourier-Stieltjes coefficients would tend to 0. Actually,

$$\int_{-\pi}^{\pi} e^{ikx} d\varphi(x) = \lim_{n \to \infty} \int_{-\pi}^{\pi} e^{ikx} d\varphi_n(x)$$

for every $k$, while obviously

$$\frac{1}{\pi} \int_{-\pi}^{\pi} e^{ikx} d\varphi_n(x) \geq 1$$

whenever $k$ is a $q_j$, for all large enough values of $n$.

§3. The assumption that the $\lambda$ of the inequality $q_{n+1}/q_n \geq \lambda$ is greater than 2 can not be replaced by the milder assumption $\lambda > 2 - \epsilon$, nor even by $\lambda \geq 2$. As a Gegenbeispiel, let $q_n = 2^{n-1}$. Then, by complete induction, we may show that

$$\prod_{k=1}^{n} (1 + \cos q_k y) = \frac{\sin^2 (2^{n-1} y)}{2^n \sin^2 (y/2)}.$$

Thus, since $\int_0^{+\infty} \frac{\sin^2 y}{y^2}\, dy = \frac{\pi}{2}$, it may be proved (as in the Fejér theorem of which, in fact, this is a special case) that

$$\phi(x) = \lim_{n\to\infty} \phi_n(x) = -x + \lim_{n\to\infty} \int_0^x \prod_{k=1}^{n} (1 + \cos q_k y)\, dy$$

$$= -x + \lim_{n\to\infty} \int_0^x \frac{\sin^2 (2^{n-1} y)}{2^n \sin^2 (y/2)}\, dy = -x + \pi \operatorname{sgn} x \pmod{2\pi}.$$

The function $\varphi(x) + x$ is monotone, but is not continuous.

§4. Bohr[10] defines a "periodenartig" function as one which is connected with a positive absolute constant $c$ in such a way that if $\epsilon$ is any positive number, there exists a translation number $\tau(\epsilon)$ pertaining to $\epsilon$ and exceeding $c$. He gives an example to show that such a function need not be almost periodic in his uniform sense. We shall show that his example is the Fourier-Stieltjes transform of a continuous but not absolutely continuous monotone function.

Bohr defines his example by the following recursion process: let $\tau_2, \tau_3, \cdots$ be a sequence of numbers such that

$$\tau_n > 2((n-1)\tau_{n-1} + (n-2)\tau_{n-2} + \cdots + 2\tau_2 + 1);\ \tau_2 > 2.$$

Put

$$f_1(x) = \begin{cases} 1 - |x| & \text{for } -1 \leqq x \leqq 1; \\ 0 & \text{for } |x| > 1; \end{cases}$$

$$f_n(x) = \sum_{m=-n-1}^{1} \frac{n-m}{n} f_{n-1}(x + m\tau_n) + f_{n-1}(x) + \sum_{m=1}^{n-1} \frac{n-m}{n} f_{n-1}(x - m\tau_n).$$

Then the desired function is

$$f(x) = \lim_{n\to\infty} f_n(x),$$

which exists uniformly on every finite interval.

[10] H. Bohr, loc. cit. 5.

For simplicity, we shall assume that the numbers $\tau_2, \tau_3, \cdots$ are integers. Let us notice that if $\varphi_n(u)$ is the Fourier transform of $f_n(x)$,

$$\varphi_1(u) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos u}{u^2};$$

$$\varphi_n(u) = \varphi_{n-1}(u) \frac{\sin^2 \frac{un\tau_n}{2}}{n \sin^2 \frac{u\tau_n}{2}}.$$

Thus

$$\varphi_n(u) = \sqrt{\frac{2}{\pi}} \frac{1 - \cos u}{u^2} \prod_{k=2}^{n} \frac{\sin^2 \frac{uk\tau_k}{2}}{k \sin^2 \frac{u\tau_k}{2}}.$$

The sum of the Fourier coefficients of $\sqrt{\frac{\pi}{2}} \frac{u^2 \varphi_n(u)}{1 - \cos u}$, which are all real and non-negative, will be

$$\prod_{k=2}^{n} \left(1 + \frac{2(k-1)}{k} + \frac{2(k-2)}{k} + \cdots + \frac{2}{k}\right) = n!.$$

On the other hand, the coefficient of $e^{iu m\tau_k}$ $(k \leq n)$ will be $(n - m)/n$.

Let us put

$$\psi_n(u) = \int_0^u \sqrt{\frac{\pi}{2}} \frac{v^2 \varphi_n(v)}{1 - \cos v} dv.$$

We shall then have

$$\psi_{n+1}(u) - \psi_n(u) = \int_0^u \sqrt{\frac{\pi}{2}} \frac{v^2 \varphi_n(v)}{1 - \cos v} \left(\sum_{m=1}^{n-1} \frac{n-m}{n} (e^{ivm\tau_n} + e^{-ivm\tau_n})\right) dv.$$

The recursive inequality between the $\tau_n$'s makes it clear that no term of this expression has a frequency less in absolute value than $\tau_n/2$. Since

$$\tau_n > 2^{n-1}(n-1)!$$

it follows that

$$|\psi_{n+1}(u) - \psi_n(u)| < \text{const.} \frac{(n+1)!}{2^{n-1}(n-1)!} < \text{const.}\ n^2 2^{-n}$$

This establishes the existence of

$$\psi(u) = \lim_{n\to\infty} \psi_n(u)$$

as a uniform limit over $-\infty < u < \infty$. The relations between the $\tau_n$'s also imply that over the range of frequencies between $-\tau_n$ and $\tau_n$, the Fourier coefficients of $\psi_n(u)$ are the same as those of $\psi(u)$. Thus the average of the coefficients of $\psi(u)$ over this range is not greater than

$$\frac{n!}{2^n(n-1)!} = \frac{n}{2^n},$$

and the average of the coefficients over the infinite range can readily be shown to be 0. Consequently $\psi(u)$ is a continuous monotone function. Finally, this function cannot be absolutely continuous, since its Fourier coefficients are not $o(1/n)$. This reasoning follows the lines laid down in §2.

By the definition of $\varphi_n(u)$,

$$f_n(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \varphi_n(u) e^{-iux}\, du.$$

Hence, by the definition of $\psi_n(u)$,

$$f_n(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1-\cos u}{u^2}\, e^{-iux}\, d\psi_n(u).$$

It follows from the continuity theorem of Lévy that

$$f(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1-\cos u}{u^2}\, e^{-iux}\, d\psi(u).$$

If then we put

$$\Phi(u) = \sqrt{\frac{\pi}{2}} \int_{-\infty}^{u} \frac{1-\cos v}{v^2}\, d\psi(v),$$

it is easy to show that $\Phi(u)$ is monotone and continuous, but not absolutely continuous. We shall have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iux}\, d\Phi(u),$$

as stated in the introduction.

§5. One of the authors[11] has introduced the following theory of the spectrum as it occurs in optics: let $f(t)$ be a measurable function defined over the interval $-\infty < t < \infty$, and let

$$F(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+x)\bar{f}(x)\,dx$$

exist for every $t$, and be a continuous function of $t$. Then we shall have

$$F(t) = \int_{-\infty}^{\infty} e^{itu}\,d\sigma(u),$$

where $\sigma(u)$ is a monotone bounded function. This function $\sigma(u)$ is called the *spectrum* of $f(t)$. The question at issue is: what functions $\sigma(u)$ are spectra of suitable functions $f(t)$? We shall prove that every bounded non-increasing function is such a $\sigma(u)$.

First, if $\sigma(u)$ is the step-function increasing by $a_n$ at the $N$ points $u_n$, respectively, we can put

$$f(t) = \sum_{1}^{N} a_n^{\frac{1}{2}} e^{iu_n t}.$$

If $\sigma(u)$ is any bounded non-increasing function, which can be thought of as normalized by putting $\sigma(-\infty) = 0$, $\sigma(\infty) = 1$, let us write

$$\sigma_n(u) = \frac{1}{n}[n\sigma(u)].$$

This function $\sigma_n(u)$ is again monotone, lies between 0 and 1, and is in addition a step-function. Thus there exists a function $f_n(t)$ which has $\sigma_n(u)$ as its spectrum. Since $f_n(t)$ is almost periodic, there will then exist a quantity $A_n > 0$ such that if $B_n > A_n$, for all $t$

$$\left| \frac{1}{2B_n} \int_{-B_n}^{B_n} f_n(x+t)\bar{f}_n(x)\,dx - \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x+t)\bar{f}_n(x)\,dx \right| < 2^{-n}.$$

Now let

$$f(t) = f_n(t) \quad \text{for} \quad \prod_{1}^{n-1} (A_k + k) \leq |t| < \prod_{1}^{n} (A_k + k)$$

and let

$$F_n(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_n(x+t)\bar{f}_n(x)\,dx$$

[11] Cf., e.g., Göttinger Nachrichten, 1925, 151–158; loc. cit. 6.

It is then an elementary result that

$$F(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(x+t)\bar{f}(x)\,dx$$

exists for every value of $t$, and is equal to

$$\lim_{n\to\infty} F_n(t) = \lim_{n\to\infty} \int_{-\infty}^{\infty} e^{itu}\,d\sigma_n(u).$$

Since, by the definition of $\sigma_n$, $\sigma_n \to \sigma$ as $n \to \infty$, we have

$$F(t) = \int_{-\infty}^{\infty} e^{itu}\,d\sigma(u),$$

by the Lévy continuity theorem. This implies that $F(t)$ is continuous. This completes the proof.

§6. In this section, we shall give an example of a function $f(t)$ with a spectrum of the Cantor type. From the point of view of existence, this adds nothing to what has been proved in the last section, but the example is interesting on its own merits. In this example, we shall give in the first instance an array $a_n$ with a spectrum of this sort. From the array theory, it is easy to derive a function $f(t)$ with a spectrum of the desired type, by putting

$$f(t) = a_{[t]}.$$

An array $a_n$ $(-\infty < n < \infty)$ is said[12] to have a spectrum if

$$b_n = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{m=-N}^{N} a_{m+n}\bar{a}_m$$

exists for every integer $n$, positive, negative, or zero. The spectrum will be given by the function

$$\sigma(u) = b_0 u + \sum_{-\infty}^{\infty}{}' \frac{b_n e^{inu}}{in},$$

which will be of limited total variation over every finite interval. In our case, we shall put

$$a_n = \prod_{\nu=2}^{\infty} 2^{\frac{1}{\nu}} \sin\left(\frac{\pi}{4} + \frac{2\pi n}{\nu!}\right).$$

[12] N. Wiener, Journ. Math. Phys. Mass. Inst. Technology 6 (1927), 145–157.

We shall show that $\{a_n\}$ has a spectrum $\sigma(u)$ which is continuous, and increases only over the set of points $S$ given by the values of

$$2\pi\left(\frac{\alpha_2}{2!}+\frac{\alpha_3}{3!}+\cdots+\frac{\alpha_\nu}{\nu!}+\cdots\right),$$

$$\text{where} \qquad \alpha_2=\pm 1,\ \alpha_3=\pm 1,\ \cdots,\ \alpha_\nu=\pm 1,\ \cdots.$$

As this set is of the Cantor type and of zero measure, the desired result follows.

Not only does the product by which $a_n$ is defined converge, but since

$$\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{2n\pi}{\nu!}\right)=1+O(n/\nu!),$$

we have

$$|a_n|\leq\prod_{\nu!<n}2^{\frac{1}{2}}\prod_{\nu!\geq n}(1+O(n/\nu!))=O(1)\prod_{\nu!<n}2^{\frac{1}{2}}=\exp O\left(\frac{\log n}{\log\log n}\right).$$

Let us put

$$a_{M_n}=\prod_{\nu=2}^{M}\sqrt{2}\sin\left(\frac{\pi}{4}+\frac{2n\pi}{\nu!}\right)=\prod_{\nu=2}^{M}\left(\frac{1-i}{2}e^{\frac{2\nu n i}{\nu!}}+\frac{1+i}{2}e^{\frac{-2\nu n i}{\nu!}}\right).$$

The frequencies of this expression are the numbers

$$B_{M_\mu}=2\pi\left(\frac{\alpha_2}{2!}+\frac{\alpha_3}{3!}+\cdots+\frac{\alpha_M}{M!}\right)$$

where each $\alpha_\nu$ is $\pm 1$, and the coefficient of $\exp(B_{M_\mu} in)$ is

$$A_{M_\mu}=\left(\frac{1-i}{2^{\frac{1}{2}}}\right)^{\alpha_2+\cdots+\alpha_M}\cdot 2^{-M/2}.$$

Since

$$\frac{1-i}{2}+\frac{1+i}{2}=1,$$

if $\mu_1$ corresponds to the sequence $\alpha_1, \cdots, \alpha_{M-1}, \alpha_M$, and $\mu_2$ to the sequence $\alpha_1, \cdots, \alpha_{M-1}, -\alpha_M$, while $\mu_3$ corresponds to the sequence $\alpha_1, \cdots, \alpha_{M-1}$, we have

$$A_{M_{\mu_1}}+A_{M_{\mu_2}}=A_{M-1,\mu_3}.$$

Furthermore, the interval $(B_{M_{\mu_1}}, B_{M_{\mu_2}})$ contains no other $B_{M_\mu}$. Thus the sum of the $A_{M_\mu}$'s corresponding to the $B_{M_\mu}$'s in the interval $(-\frac{1}{2}, x)$

contains not more than one term of absolute value $2^{-\frac{1}{2}}$, not more than one of absolute value $2^{-1}$, and so on; or in other words can not exceed $(1 - 2^{-\frac{1}{2}})^{-1}$ in modulus. An argument of exactly the same type will show that if we consider the interval $(x, y)$, where

$$|x - y| \leq \frac{1}{\nu!} + \frac{1}{(\nu+1)!} + \cdots,$$

the sum of the $A_{M_\mu}$'s corresponding to the $B_{M_\mu}$'s in this interval can not exceed

$$\frac{2^{\frac{\nu}{2}+1}}{2^{\frac{1}{2}}-1}$$

in modulus.

Let it be noted that this bound does not depend on $M$. Let it also be noted that for all points $x$ in any interval between intervals of the form

$$\frac{\alpha_2}{2!} + \frac{\alpha_3}{3!} + \cdots + \frac{\alpha_M}{M!} \pm \sum_{M+1}^{\infty} \frac{1}{\nu!},$$

the sum of the $A_{N_\mu}$'s corresponding to $B_{N_\mu}$'s in the interval $(-\frac{1}{2}, x)$ is independent of $N$ for $N \geq M$. If we write $\varphi_N(x)$ for this sum, we thus see that it converges uniformly to a limit $\varphi(x)$ as $N \to \infty$. Thus for fixed $\epsilon$, but uniformly in $x$,

$$\lim_{M\to\infty} \frac{1}{2\epsilon} \int_{-\frac{1}{2}}^{x} |\varphi_M(y+\epsilon) - \varphi_M(y-\epsilon)|^2 dy = \frac{1}{2\epsilon} \int_{-\frac{1}{2}}^{x} |\varphi(y+\epsilon) - \varphi(y-\epsilon)|^2 dx.$$

Now let

$$\frac{1}{N!} - 2\sum_{N+1}^{\infty} \frac{1}{\nu!} > 2\epsilon \geq \frac{1}{(N-1)!} - 2\sum_{N}^{\infty} \frac{1}{\nu!}.$$

If we notice that

$$\frac{1-i}{2} + \frac{1+i}{2} = 1; \qquad \left(\frac{1-i}{2}\right)^2 + \left(\frac{1+i}{2}\right)^2 = 1;$$

we shall see that except over a set of points not exceeding

$$2^{N+4} \sum_{N+3}^{\infty} \frac{1}{\nu!} = O(2^{N+4}(N+2)^{-N-2})$$

in measure, we shall have

$$|\varphi(x+\epsilon)-\varphi(x-\epsilon)|^2=\sum|\varphi_N(A_{N\mu}+0)-\varphi_N(A_{N\mu}-0)|^2,$$

the sum being taken over all discontinuities of $\varphi_N$ in $(x-\epsilon, x+\epsilon)$. Elsewhere our estimate of the sums of the $A_{N\mu}$'s may be used to give an upper estimate of $|\varphi(x+\epsilon)-\varphi(x-\epsilon)|$. Let us notice that

$$|\varphi_N(A_{N\mu}+0)-\varphi_N(A_{N\mu}-0)|^2=2^{-N}.$$

From this it follows that if $2\pi x$ lies in one of the intervals belonging to $\bar{S}$, then the double limit

$$\lim_{\substack{\epsilon\to 0\\ N\to\infty}}\frac{1}{2\epsilon}\int_{-\frac{1}{2}}^{x}|\varphi_N(y+\epsilon)-\varphi_N(y-\epsilon)|^2dy$$

exists. It is now not hard to prove that it will also exist for every $x$ in $(-\frac{1}{2}, \frac{1}{2})$. Here we use the fact of the monotoneness of the approximations, and the fact that their maximum jumps tend to zero.

Now let us put

$$\sigma(u)=2\pi\lim_{\substack{\epsilon\to 0\\ N\to\infty}}\frac{1}{2\epsilon}\int_{-\frac{1}{2}}^{u/2\pi}|\varphi_N(y+\epsilon)-\varphi_N(y-\epsilon)|^2dy$$

and let us take $\varphi_N(u)$ to be $2^{-N+1}\pi$ times the number of $A_{N\mu}$'s in $(-\frac{1}{2}, u/2\pi)$. We shall have

$$\sigma(u)=\lim_{N\to\infty}\sigma_N(u).$$

We shall easily be able to establish that

$$2\pi\lim_{\substack{\epsilon\to 0\\ N\to\infty}}\frac{1}{2\epsilon}\int_{-\frac{1}{2}}^{\frac{1}{2}}|\varphi_N(y+\epsilon)-\varphi_N(y-\epsilon)|^2\cos 2m\pi x\,dx=\int_{-\pi}^{\pi}\cos mu\,d\sigma(u).$$

Now,

$$a_{Nn}=\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi nix}d\varphi_N(x).$$

Thus

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi inx}d\{\varphi_N(x+\epsilon)-\varphi_N(x-\epsilon)\}=a_{Nn}(e^{-2\pi in\epsilon}-e^{2\pi in\epsilon}),$$

if we define $\varphi(x)$ to be of period 1. Hence, on integrating by parts,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}(\varphi_N(x+\epsilon)-\varphi_N(x-\epsilon))e^{2\pi inx}dx=a_{Nn}\frac{\sin 2\pi n\epsilon}{n\pi}.$$

It follows that

$$\int_{-\pi}^{\pi} \cos mu \, d\sigma(u) = \lim_{\substack{M\to\infty \\ \epsilon\to 0}} \frac{\pi}{\epsilon} \sum_{-\infty}^{\infty} a_{M,m+n}\bar{a}_{Mn} \frac{\sin^2 2\pi n\epsilon}{n^2\pi^2}$$

$$= \lim_{\epsilon\to 0} \frac{\pi}{\epsilon} \sum_{-\infty}^{\infty} a_{m+n}\bar{a}_n \frac{\sin^2 2n\pi\epsilon}{n^2\pi^2}.$$

By a Tauberian theorem,[13] it follows that

$$\int_{-\pi}^{\pi} \cos mu \, d\sigma(u) = 2\pi \lim_{N\to\infty} \frac{1}{2N+1} \sum_{-N}^{N} a_{m+n}\bar{a}_n = 2\pi b_m,$$

and the array $\{a_n\}$ has a spectrum. We have

$$b_0 x + \sum_{-N}^{N}{}' \frac{b_n e^{inx}}{in} = b_0 x + 2\sum_{1}^{N} \frac{b_n}{n} \sin nx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left\{ x + 2\sum_{1}^{N} \frac{\cos n\xi \sin nx}{n} \right\} d\sigma(\xi)$$

$$= \frac{1}{2\pi} \left\{ x + 2\sum_{1}^{N} \frac{\sin nx}{n} \right\} \{\sigma(\pi) - \sigma(-\pi)\}$$

$$+ \frac{1}{\pi} \int_{-\pi}^{\pi} \sigma(\xi) \sum_{1}^{N} \sin n\xi \sin nx \, d\xi.$$

The first part of this converges to 0, and the second part converges in the mean to $\sigma(u)$, which we have already seen to be a singular function.

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[13] N. Wiener, loc. cit. 6, §5.

# A GENERALIZATION OF IKEHARA'S THEOREM

By N. Wiener and H. R. Pitt

1. Suppose that $s(y)$ is measurable, $s(y) \geq -K$, and that $s(y)e^{-\sigma y}$ is integrable in $(0, \infty)$ for any positive $\sigma$. Let $\omega = \sigma + it$,

$$S(\omega) = \int_0^\infty e^{-\omega y} s(y)\, dy. \tag{1.1}$$

Let $B > 0$. If $0 < \alpha \leq 1$, we denote by $C_\alpha$ the curve

$$|t| = B\sigma^\alpha, \qquad \sigma \geq 0, \tag{1.2}$$

and denote the $y$-axis by $C_0$.

One of the conditions in our main theorem in that $S(\omega)$ is integrable along $C_\alpha$. We first explain what this involves. If $0 < \alpha \leq 1$, we suppose that $S(\omega)$ is absolutely integrable on $C_\alpha$ and that we have the usual relation

$$S(\omega) = \frac{1}{2\pi i} \int_{C_\alpha} \frac{S(z)\, dz}{z - \omega} \tag{1.3}$$

for points $\omega$ inside $C_\alpha$. If $\alpha = 0$, we suppose that $S(\omega)$ can be defined for $\sigma = 0$ in such a way that for any positive $A$

$$S(\omega) = \frac{1}{2\pi i} \int_{C'} \frac{S(z)\, dz}{z - \omega}, \tag{1.4}$$

where $C'$ consists of the three lines $t = \pm A$, $\sigma \geq 0$ and $|t| \leq A$, $\sigma = 0$.

Our main object is to prove the following result.

Theorem 1. *Suppose that*

$$A > 0, \qquad 0 \leq \alpha < 1, \tag{1.5}$$

$$s(y) \geq -K, \tag{1.6}$$

*and that $S(\omega)$, defined as above, is integrable on $C_\alpha$. Then*

$$\lim_{x \to \infty} x^{-\alpha} \int_{x - Ax^\alpha}^{x + Ax^\alpha} s(y)\, dy = 0.$$

We prove the theorem first in the extreme cases $\alpha = 0$, $\alpha = 1$, and then show how these are related to known results. In particular, we deduce Ikehara's theorem[1] from the case $\alpha = 0$.

We may plainly suppose that $s(y) = 0$ for $y \leq 1$. Then

$$|S(\sigma + it)| \leq \int_1^\infty e^{-\sigma y} |s(y)|\, dy \leq e^{-\frac{1}{2}\sigma} \int_1^\infty e^{-\frac{1}{2}\sigma y} s(y)\, dy \tag{1.7}$$

$$= 0(e^{-\frac{1}{2}\sigma}) \quad \text{as} \quad \sigma \to \infty.$$

This is enough to ensure that (1.3) and (1.4) are true, by Cauchy's theorem, when $S(\omega)$ is continuous on $C_\alpha$ at the origin.

2. Lemma 1. *If $s(y) \geq -K$ and $S(\omega)$ is integrable on $C_1$, then*

$$\lim_{x \to \infty} x^{-1} \int_0^x s(y)\, dy = 0.$$

[Since $s(y) = 0$ for $y \leq 0$, this is plainly equivalent to the case $\alpha = 1$ of the theorem.]

We have, using (1.3),

$$S(\sigma) = \frac{1}{2\pi i} \int_{C_1} \frac{S(\omega)\, d\omega}{\omega - \sigma},$$

$$|\sigma S(\sigma)| \leq \frac{1}{2\pi} \int_{C_1} \left| \frac{S(\omega)\, d\omega}{\omega - \sigma} \right| = 0(1) \qquad (\sigma \to 0),$$

since $\sigma\,|\omega - \sigma|^{-1}$ is bounded on $C_1$ and tends to 0 as $\sigma \to 0$ at each point of $C_1$ except the origin. Since $s(y) \geq -K$, the conclusion follows from a well known theorem[2] of Hardy and Littlewood.

Lemma 2. *If $A > 0$, $s(y) \geq -K$ and $S(\omega)$ is integrable on $C_0$, then*

$$\lim_{x \to \infty} \int_{x-A}^{x+A} s(y)\, dy = 0.$$

If $\lambda > 0$, we can write[3]

$$\int e^{-ity} h_\lambda(x - y)\, dy = \left(1 - \frac{|t|}{\lambda}\right) e^{-itx} \qquad (|t| \leq \lambda),$$

$$= 0 \qquad (|t| > \lambda),$$

[1] S. Ikehara, Journal of Math. and Phys., M. I. T. (1931), 1–12. S. Bochner, "*Ein Satz von Landau und Ikehara.*" Math. Zeitschrift 37 (1933), 1–9.

[2] G. H. Hardy and J. E. Littlewood, "*Tauberian Theorems concerning power series and Dirichlet series whose coefficients are positive.*" Proc. London Math. Soc., 13 (1913), 174–198. We use an extension of Theorem 8 with $L(x) = 1$, $\alpha = 1$.

[3] Integrals in which the limits are omitted are over the range $(-\infty, \infty)$.

where

$$h_\lambda(x) = \frac{\sin^2 \lambda x}{\pi \lambda x^2}.$$

Also

$$\int e^{-ity} e^{-\sigma y} s(y)\, dy = S(\sigma + it),$$

and it follows from the Fourier transform analogue of Parseval's theorem that

$$\int h_\lambda(x-y) e^{-\sigma y} s(y)\, dy = \int_{-\lambda}^{\lambda} \left(1 - \frac{|t|}{\lambda}\right) e^{-itx} S(\sigma + it)\, dt. \tag{2.1}$$

We take $C'$ to be the three lines $t = \pm 2\lambda$, $\sigma \geq 0$, and $|t| \leq 2\lambda$, $\sigma = 0$. Then it follows from (1.4) and the fact that $S(\omega)$ is absolutely integrable on $C'$ that

$$\frac{1}{2\pi i} \int_{C'} \frac{S(z)\, dz}{z - \omega} = 0$$

if $z$ is outside $C'$. Hence, if $|t| \leq \lambda$,

$$S(\sigma + it) = \frac{1}{2\pi i} \int_{C'} \frac{S(z)\, dz}{z - \sigma - it}, \qquad 0 = \frac{1}{2\pi i} \int_{C'} \frac{S(z)\, dz}{z + \sigma - it},$$

and we obtain by subtraction

$$S(\sigma + it) = \frac{1}{2\pi} \int_{-2\lambda}^{2\lambda} \frac{2\sigma S(iy)\, dy}{(y-t)^2 + \sigma^2} + \epsilon(\sigma, t),$$

where

$$|\epsilon(\sigma, t)| \leq \frac{2\sigma}{\pi(\lambda^2 - \sigma^2)} \int_{C'} |S(z)|\, dz.$$

It is easy to deduce from this that

$$\lim_{\sigma \to 0} \int_{-\lambda}^{\lambda} |S(\sigma + it) - S(it)|\, dt = 0,$$

and therefore, using the fact that $s(y) + K \geq 0$ in (2.1), we have

$$\int h_\lambda(x-y) s(y)\, dy = \int_{-\lambda}^{\lambda} \left(1 - \frac{|t|}{\lambda}\right) e^{-itx} S(it)\, dt.$$

By the Riemann-Lebesgue theorem, it follows that

$$\lim_{x \to \infty} \int h_\lambda(y) s(x-y)\, dy = 0. \tag{2.2}$$

Now let

$$\epsilon > 0, \quad g_\epsilon(x) = \int_{x-\epsilon}^{x+\epsilon} s(y)\,dy.$$

Since (2.2) holds for any positive $\lambda$, and since $h_\lambda(x) \geq \lambda/\pi^2$ for $|x| \leq 1/\lambda$, it follows that $g_\epsilon(x)$ is bounded and

$$\overline{\lim_{x\to\infty}} |g_\epsilon(x)| \leq \eta(\epsilon), \qquad \lim_{\epsilon\to 0} \eta(\epsilon) = 0. \tag{2.3}$$

On integrating (2.2) between $x - A$ and $x + A$, we have

$$\lim_{x\to\infty} \int h_\lambda(y) g_A(x-y)\,dy = 0.$$

Now $g_A(x)$ is bounded for fixed $A$, and

$$\lim_{x\to\infty} |g_A(x-y) - g_A(x)| \leq 2\eta(y).$$

Hence

$$\overline{\lim_{x\to\infty}} \left| g_A(x) - \int h_\lambda(y) g_A(x-y)\,dy \right| \leq \overline{\lim_{x\to\infty}} \int h_\lambda(y) |g_A(x-y) - g_A(x)|\,dy$$

$$\leq 2\int h_\lambda(y)\eta(y)\,dy = 2\int h_1(y)\eta(y/\lambda)\,dy.$$

On letting $\lambda \to \infty$, it follows from (2.3) that the left hand side is zero, and hence

$$\lim_{x\to\infty} g_A(x) = 0.$$

Ikehara's theorem can be stated as follows.

THEOREM 2. *Suppose that $k(y)$ is non-decreasing in $(0, \infty)$ and that*

$$f(\omega) = \int_0^\infty e^{-\omega y}\,dk(y)$$

*converges absolutely for $\sigma > 1$. Suppose that $f(w) - (w-1)^{-1}$ is continuous to the right of $\sigma = 1$. Then*

$$\lim_{y\to\infty} e^{-y} k(y) = 1.$$

Let $s(y) = e^{-y}k(y) - 1$, so that $s(y) \geq -1$. We have, by partial integration,

$$\int_0^\infty e^{-\omega y} s(y)\,dy = \int_0^\infty e^{-(\omega+1)y} k(y)\,dy - \int_0^\infty e^{-\omega y}\,dy = \frac{f(\omega+1)}{\omega+1} - \frac{1}{\omega},$$

and this is continuous to the right of the $y$-axis. Therefore, by Lemma 2,

$$\lim_{x\to\infty}\int_{x-A}^{x+A} e^{-y}k(y)\,dy = 2A$$

for any positive $A$. But since $k(y)$ is monotone,

$$e^{2A}k(x+A)e^{-(x+A)} \geq e^{-y}k(y) \geq e^{-2A}k(x-A)e^{-(x-A)}$$

for $|x-y| \leq 2A$. Therefore, as $x \to \infty$,

$$e^{2A}\,\underline{\lim}\, k(x)e^{-x} \geq 1 \geq e^{-2A}\,\overline{\lim}\, k(x)e^{-x},$$

and we obtain the required conclusion by letting $A \to 0$.

3. *Proof of Theorem* 1 *for* $0 < \alpha < 1$.

We write $C$ instead of $C_\alpha$ and put $\alpha^{-1} = a$. Our hypothesis implies that $S(|t|^a + it)$ is absolutely integrable in $(-\infty, \infty)$. We first show that

$$\lim_{\epsilon\to 0}\int e^{ixt}S(\epsilon + |t|^a + it)\,dt = \int e^{ixt}S(|t|^a + it)\,dt. \tag{3.1}$$

We have, by (1.3)

$$S(\epsilon + |t|^a + it) = \frac{1}{2\pi i}\int_C \frac{S(\omega)\,d\omega}{\omega - \epsilon - |t|^a - it},$$

$$0 = \frac{1}{2\pi i}\int_C \frac{S(\omega)\,d\omega}{\omega + \epsilon - |t|^a - it},$$

and on subtracting we get

$$S(\epsilon + |t|^a + it) = \frac{\epsilon}{\pi i}\int_C \frac{S(\omega)\,d\omega}{(\omega - |t|^a - it)^2 - \epsilon^2}.$$

Let $\eta$ be positive and independent of $\epsilon$. Then

$$\begin{aligned}\int_{-\eta}^{\eta} e^{ixt}S(\epsilon + |t|^a + it)\,dt &= \int_{-\eta}^{\eta} e^{ixt}dt\,\frac{\epsilon}{2\pi i}\int_C \frac{S(\omega)\,d\omega}{(\omega - |t|^a - it)^2 - \epsilon^2} \\ &= \frac{1}{2\pi i}\int_C S(\omega)\,d\omega \int_{-\eta}^{\eta} \frac{\epsilon e^{ixt}dt}{(\omega - |t|^a - it)^2 - \epsilon^2}.\end{aligned} \tag{3.2}$$

Since $\alpha < 1$, we can choose $\eta > 0$ so that

$$(3\eta)^{2a-1}a^2 \leq \tfrac{1}{2}. \tag{3.3}$$

Then if $\omega = |\tau|^a + i\tau$, $|\tau| \leq 2\eta$, we have

$$\big||\tau|^a - |t|^a\big| \leq \big||\tau| - |t|\big|\,(3\eta)^{a-1}a \leq |\tau - t|\,(3\eta)^{a-1}a,$$

by the second mean-value theorem, and it follows that

$$|(\omega - |t|^\alpha - it)^2 - \epsilon^2| = |[(|\tau|^\alpha - |t|^\alpha) + i(\tau - t)]^2 - \epsilon^2|$$
$$\geq (\tau - t)^2 + \epsilon^2 - a^2(3\eta)^{2\alpha-2}(\tau - t)^2 \geq \tfrac{1}{2}(\tau - t)^2 + \epsilon^2.$$

Hence, if $|\tau| \leq 2\eta$,

$$\left|\int_{-\eta}^{\eta} \frac{\epsilon e^{ixt}\,dt}{(\omega - |t|^\alpha - it)^2 - \epsilon^2}\right| \leq 2\epsilon \int \frac{dt}{(\tau - t)^2 + 2\epsilon^2} \tag{3.4}$$
$$= 2\int \frac{dt}{t^2 + 2} < 6.$$

If $|\tau| \geq 2\eta$, we have $|\tau - t| \geq \eta$, $||\tau|^\alpha - |t|^\alpha| \geq \eta^\alpha$, and

$$|(\omega - |t|^\alpha - it)^2 - \epsilon^2| \geq 2|\tau - t|\,||\tau|^\alpha - |t|^\alpha| \geq 2\eta^{1+\alpha}.$$

For such values of $\tau$,

$$\left|\int_{-\eta}^{\eta} \frac{\epsilon e^{ixt}\,dt}{(\omega - |t|^\alpha - it)^2 - \epsilon^2}\right| \leq \epsilon\eta^{-\alpha} = 0(1) \qquad (\epsilon \to 0). \tag{3.5}$$

Combining (3.2), (3.4) and (3.5), we obtain

$$\overline{\lim_{\epsilon\to 0}}\left|\int_{-\eta}^{\eta} e^{ixt} S(\epsilon + |t|^\alpha + it)\,dt\right| \leq \frac{3}{\pi}\int_{-\eta}^{\eta} |S(|t|^\alpha + it)|\,dt. \tag{3.6}$$

Now

$$\overline{\lim_{\epsilon\to 0}}\left|\int e^{ixt} S(|t|^\alpha + it)\,dt - \int e^{ixt} S(\epsilon + |t|^\alpha + it)\,dt\right|$$

$$\leq \overline{\lim_{\epsilon\to 0}}\left|\int_{-\infty}^{-\eta} + \int_{\eta}^{\infty}\right| S(|t|^\alpha + it) - S(\epsilon + |t|^\alpha + it)\Big|\,dt \tag{3.7}$$

$$+ \overline{\lim_{\epsilon\to 0}}\left|\int_{-\eta}^{\eta} e^{ixt} S(\epsilon + |t|^\alpha + it)\,dt\right| + \int_{-\eta}^{\eta} |S(|t|^\alpha + it)|\,dt.$$

It follows from (1.7) that $S(\epsilon + |t|^\alpha + it)$ is majorized in the range $|t| \geq \eta$ by an integrable function which is independent of $\epsilon$, and since $S(\omega)$ is uniformly continuous for $\sigma \geq \eta^\alpha$, the first term on the right of (3.7) vanishes. Using (3.6), it is plain that the sum of the remaining terms does not exceed

$$\left(1 + \frac{3}{\pi}\right)\int_{-2\eta}^{2\eta} |S(|t|^\alpha + it)|\,dt.$$

Since this is arbitrarily small by choice of $\eta$, we have established (3.1).

Now using (3.1), we have

$$\begin{aligned}\int e^{ixt}S(|t|^a + it)\,dt &= \lim_{\epsilon\to 0}\int e^{ixt}S(\epsilon + |t|^a + it)\,dt \\ &= \lim_{\epsilon\to 0}\int e^{ixt}\,dt\int s(y)e^{-(\epsilon+|t|^a+it)y}\,dy \\ &= \lim_{\epsilon\to 0}\int s(y)e^{-\epsilon y}\,dy\int e^{-|t|^a y+i(x-y)t}\,dt \\ &= \lim_{\epsilon\to 0}\int s(y)e^{-\epsilon y}y^{-a}F[(x-y)y^{-a}]\,dy,\end{aligned} \tag{3.8}$$

where

$$F(u) = \int e^{iut}e^{-|t|^a}\,dt = 2\int_0^\infty e^{-t^a}\cos ut\,dt. \tag{3.9}$$

We now consider two cases, according as $a$ is not an even integer, or is an even integer. In the first case, if we put $\nu = [a]$ the function

$$e^{-|t|^a} + |t|^a e^{-|t|}$$

has $\nu + 2$ successive derivatives belonging to $L$ over $(-\infty, \infty)$. We shall have

$$\int_{-\infty}^{\infty}(e^{-|t|^a} + |t|^a e^{-|t|})e^{iut}\,dt = O(u^{-\nu-2}) = o(u^{-a-1}), \tag{3.10}$$

and

$$\begin{aligned}\int_{-\infty}^{\infty}|t|^a e^{-|t|}e^{iut}\,dt &= \int_0^\infty t^a e^{-t(1+iu)}\,dt + \int_0^\infty t^a e^{-t(1-iu)}\,dt \\ &= 2\Gamma(a+1)R[(1+iu)^{-a-1}].\end{aligned} \tag{3.11}$$

Accordingly, unless $a$ is an even integer,

$$F(u) = \Omega(u^{-a-1}) = O(u^{-a-1}), \tag{3.12}$$

and $F(u)$ is free from zeros for $u >$ some $u_0$. Moreover, $F(u)$ is continuous and $O(u^{-2})$ for large values of $u$, and is therefore integrable in $(-\infty, \infty)$.

Since $s(y) + K \geq 0$ and $e^{-\epsilon y}$ increases as $\epsilon \to 0$, it follows that $s(y)y^{-a}F[(x-y)y^{-a}]$ is integrable, and by (3.8)

$$\begin{aligned}\int s(y)y^{-a}F[(x-y)y^{-a}]\,dy &= \lim_{\epsilon\to 0}\int s(y)e^{-\epsilon y}y^{-a}F[(x-y)y^{-a}]\,dy \\ &= \int e^{ixt}S(|t|^a + it)\,dt.\end{aligned}$$

Therefore, by the Riemann-Lebesgue theorem,

$$\lim_{x\to\infty} \int s(y)y^{-a}F[(x-y)y^{-a}]\,dy = 0. \tag{3.13}$$

In the case in which $a$ is an even integer, it is a known fact[4] that there exists a $\theta > 0$, such that

$$F(u) = 2\int_0^\infty e^{-t^a}\cos ut\,dt = O\left(\exp\left(-\theta\,|\,u\,|^{\frac{a}{a-1}}\right)\right).$$

It follows that for each $x$,

$$F[(x-y)y^{-a}] = O(\exp(-\vartheta y)),$$

and hence, by dominated convergence, that

$$\int s(y)y^{-a}F[(x-y)y^{-a}]\,dy = \lim_{\epsilon\to 0}\int s(y)e^{-\epsilon y}y^{-a}F[(x-y)y^{-a}]\,dy$$

Thus (3.13) is again valid, and if we put $\gamma = (1-\alpha)^{-1} = a/(a-1)$, we can change our variables $x$ and $y$, and establish

$$\lim_{x\to\infty}\int_0^\infty s(y^\gamma)F[(x^\gamma - y^\gamma)y^{1-\gamma}]\,dy = 0. \tag{3.14}$$

Let us notice that since $\gamma > 1$, if $x > 0$, $y > 0$, we have

$$|\,(x^\gamma - y^\gamma)y^{1-\gamma}\,| = |\,y\,|\left|\left(\frac{x}{y}\right)^\gamma - 1\right| \geq |\,y\,|\left|\frac{x}{y} - 1\right| = |\,x-y\,|.$$

Furthermore,

$$F[(x^\gamma - y^\gamma)y^{1-\gamma}] = F[(y^\gamma + \gamma y^{\gamma-1}(x-y) + O(x-y)^2 y^{\gamma-2}y^{1-\gamma} - y]$$
$$= F\left[\gamma(x-y) + \frac{O(x-y)^2}{y}\right].$$

Thus there exists a function $F^*(x, y)$, defined over $(-\infty, \infty)$ in both arguments; with the properties

$$F^*(x, y) \leq \limsup_{u>|x-y|} F(u),$$

and for all $B$, we have uniformly in $u$ over $-B < x < B$,

$$\lim F^*(x+u, u) = F(x);$$

[4] Titchmarsh, "Theory of Functions," 8-47.

and such that

$$\lim_{x\to\infty}\int_0^\infty s(y^\gamma)F^*(x,y)\,dy = 0.$$

In the case in which $a$ is not an even integer, we have

$$F^*(x,y) = O(|x-y|^{-a-1})$$

and in the even integer case,

$$F^*(x,y) = O(\exp(-\vartheta\,|x-y|^\gamma).$$

In the first case,

$$\int_{-\infty}^{\infty}|F^*(x,y)|\,(A+|x-z|^{a+1})^{-1}dz = O(|x-y|^{-a-1}),$$

and in the second case,

$$\int_{-\infty}^{\infty}|F^*(x,y)|\exp(-\eta|x-z|^\gamma)\,dz = O\left(\exp\left(-\frac{\eta}{2^\gamma}|x-y|^\gamma\right)\right)$$

$$(0<\eta<\gamma).$$

In either case, we shall thus have

$$\lim_{x\to\infty}\int_0^\infty s(y^\gamma)Q(x,y)\,dy = 0, \tag{3.15}$$

where $Q(x,y)$ is defined in the first case by

$$Q(x,y) = \int_{-\infty}^{\infty}F^*(z,y)(A+|x-z|^{a+1})^{-1}dz,$$

and in the second case by

$$Q(x,y) = \int_{-\infty}^{\infty}F^*(z,y)\exp(-\eta|x-z|^\gamma)\,dz.$$

We shall now show that if $A$ is large enough, or $\eta$ small enough, $Q(x,y)$ will be positive, will have an integral with respect to $y$ which is bounded in $x$, and over the range $-\infty < y < \infty$, will exceed a fixed positive constant for large enough values of $x$.

$$Q(x,y) = Q_1(x,y) + Q_2(x,y) + Q_3(x,y)$$

where

$$Q_1(x, y) = \int_{-\infty}^{y-B} F^*(z, y)(A + |x - z|^{\alpha+1})^{-1} dz;$$

$$Q_2(x, y) = \int_{y-B}^{y+B} F^*(z, y)(A + |x - z|^{\alpha+1})^{-1} dz;$$

$$Q_3(x, y) = \int_{y+B}^{\infty} F^*(z, y)(A + |x - z|^{\alpha+1})^{-1} dz$$

As $x$ becomes positively infinite, we shall have uniformly in $y$,

$$Q_2(x, y) \sim R(x - y),$$

where

$$R(x) = \int_{-B}^{B} F(\gamma z)(A + |x - z|^{\alpha+1})^{-1} dz$$

$$= (A + |x|^{\alpha+1})^{-1} \int_{-B}^{B} F(\gamma z)\, dz + \int_{-B}^{B} F(\gamma z) \frac{|x|^{\alpha+1} - |x - z|^{\alpha+1}}{(A + |x|^{\alpha+1})(A + |x - z|^{\alpha+1})} dz$$

$$= (A + |x|^{\alpha+1})^{-1} \int_{-B}^{B} F(\gamma z)\, dz + \vartheta \frac{B^\alpha}{A} (A + |x|^{\alpha+1})^{-1} \qquad (|\vartheta| < \text{const.}).$$

On the other hand,

$$|Q_1(x, y)| \leq \text{const.} \int_{-\infty}^{-B} |z|^{-\alpha-1}(A + |(x - y) - z|^{\alpha+1})^{-1} dz.$$

If we take the part of this integral corresponding to $|x - y - z|^{\alpha+1} > A$, it can not exceed

$$\text{const.}(A + |x - y|^{\alpha+1})^{-1} B^{-\alpha}.$$

If on the other hand, we take the part for which $|x - y - z|^{\alpha+1} < A$, it can not exceed

$$\text{const. } A^{-1-(\alpha+1)^{-1}} \cdot \min\,(B^{-\alpha-1}, (A^{-(A+1)} - |x - y|)^{-\alpha-1}),$$

where $\min(u, v)$ signifies the smaller of the two quantities, $u$ and $v$, or their common value if they coincide. If now $B = A^{\frac{1}{\alpha+1}}$, we see that

$$|Q_1(x, y)| \leq \text{const.}(A + |x - y|^{\alpha+1})^{-1} B^{-\alpha}.$$

Similarly

$$| Q_3(x, y) | \leq \text{const.}(A + | x - y |^{a+1})^{-1}B^{-a}.$$

Let us notice that as $A$ and consequently $B$ tends to infinity,

$$B^a/A = A^{-1/(a+1)} \to 0,$$

and

$$\int_{-B}^{B} F(\gamma z)\, dz \to \int_{-\infty}^{\infty} F(\gamma z)\, dz = \frac{2\pi}{\gamma} > 0.$$

Hence, if we combine $Q_1$, $Q_2$, and $Q_3$, we see that for large enough $A$ independent of $x$, for all sufficiently large $x$, we have for all $y$,

$$Q(x, y) > \left(\frac{2\pi}{\gamma} - \epsilon\right)(A + | x |^{a+1})^{-1}.$$

We also can prove at once that

$$\int_{-\infty}^{\infty} Q(x, y)\, dy < \text{const.}$$

In the case in which $a$ is an even integer, we can give a similar proof that $\eta$ may be selected once for all, so that $Q(x, y)$ will be positive for all $y$'s, for sufficiently large values of $x$, and that the last formula will hold good. As in the first case, for large enough $x$, the lower bound of the value of $Q(x, y)$ over $x \leq y \leq x + 1$ will be positive. Remembering that $s(y)$ is bounded below, we shall have by (3.15), for some constant $H$,

$$\int_{x}^{x+1} s(y^\gamma)\, dy \leq H. \tag{3.16}$$

It is now easy to show that

$$\lim_{A \to \infty} \left\{ \sum_{n=-\infty}^{\infty} \overline{Bd}_{n \leq s \leq n+1} | F(x^\gamma - y^\gamma)y^{1-\gamma}] - F(\gamma(x - y)) | \right\} = 0.$$

and combining this with (3.14) and (3.16), we obtain

$$\lim_{x \to \infty} \int s(y^\gamma)F(\gamma(x - y))\, dy = 0.$$

It follows now from a known extension[5] of Wiener's general Tauberian theorem, since the Fourier transform of $F(x)$ is never zero, that

$$\lim_{x\to\infty} \int_{x-A}^{x+A} s(y^\gamma)\, dy = 0. \tag{3.17}$$

Finally, we have

$$\lim_{x\to\infty} x^{-\alpha} \int_{x-Ax^\alpha}^{x+Ax^\alpha} s(y)\, dy = \lim_{x\to\infty} \gamma x^{-\alpha} \int_{x^{1-\alpha}-A+\theta(1)}^{x^{1-\alpha}+A+\theta(1)} s(y^\gamma) y^{\gamma-1} dy$$

$$= \lim_{x\to\infty} \gamma \int_{x^{1-\alpha}-A+\theta(1)}^{x^{1-\alpha}+A+\theta(1)} s(y^\gamma)\, dy = \lim_{x\to\infty} \gamma \int_{x^{1-\alpha}-A}^{x^{1-\alpha}+A} s(y^\gamma)\, dy$$

since $y^{\gamma-1}x^{-\alpha} \to 1$ uniformly in the range of integration, and $s(y) \geq -K$. The conclusion now follows from (3.17).

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[5] H. R. Pitt, *"A Remark on Wiener's General Tauberian Theorem."* Duke Mathematical Journal, 4 (1938).